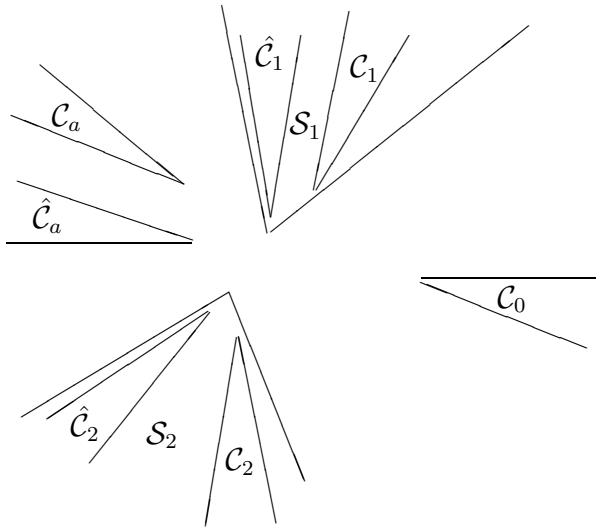
Fractional Statistics in Algebraic Quantum Field Theory and the Fractional Quantum Hall-Effect

An Application of the P_1CT Theorem



Theoretical Physics Master Thesis

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Introduction

This report describes the connection of the Fractional Quantum Hall Effect with Braid statistics in Algebraic Quantum Field Theory. It tries to bridge the gap between the Laughlin's incompressible quantum fluid model for the FQHE and the Duality assumption proposed by J.Fröhlich, F. Gabbiani and P.A. Marchetti an important pile in between is Chern-Simons Theory.
Chapter 1 introduces Quantum Hall Effects and the systems in which they can be found, it furthermore gives the theory of the Integral Quantum Hall Effect which is used to introduce the single electron states in an effectively two dimensional system. In this chapter the stabilization of the integer effect by impurities is explained. The Integral QHE is described to give the reader the opportunity to compare it to the Fractional Quantum Hall Effect.
Chapter 2 is devoted to the theory of the incompressible quantum fluid proposed by R.B. Laughlin to describe the Fractional Quantum Hall Effect. Generalizations are described and the nature of the excitations is discussed. Furthermore it is tried to extract the essential features needed for an analysis within the framework of gauge-theory.
Chapter 3 describes a $U(1) \times SU(2)$ gauge theory for generalized Hall-effects proposed by J. Fröhlich and U.M. Studer, where we concentrate on the $U(1)$ part of the theory describing the Fractional Quantum Hall Effect. Assuming strong clusterproperties and taking the scaling limit this theory gives rise to a Chern-Simons theory. Fröhlich and Studer assume that strong clusterproperties model the energy gap present in Quantum Hall-systems.
Chapter 4 connects the assumptions used in chapter 3 to the essential features we extracted from Laughlin's Theory. Using P_1CT-symmetry in $2+1$-dimensions we are able to proof a theorem of Borchers originally proved in $3+1$-dimensions also in $2+1$-dimensions. This enables us to connect uniqueness of the vacuum-state with strong-cluster properties for Bose-fields. In order to apply this theorem to Quantum Hall-effects it should be generalized to currents constructed in field algebra's satisfying fractional statistics.
Chapter 5 contains a short discussion of duality in its original form as proposed by R.Haag and duality as proposed by J. Fröhlich, F.Gabbiani and P.A. Marchetti based on the properties of the Abelian Higgs-model with a Chern-Simons term added. Furthermore it is sketched how the proposed form of duality leads to Braid statistics.

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Jeroen Nijhof always helped with LaTeX and computer problems.

Chapter 1

The Quantum Hall-effect

This chapter intoduces the quantum Hall-effect and contains in addition a short introduction to the theory of the Integer Quantum Hall-effect, most of the information for this chapter we found in [PG87] and [Sto92]. The theory of the Fractional Quantum Hall-effect is discussed separately in the next chapters.

1.1 Quantum Hall-effects

In 1980 K. von Klitzing discovered the Quantum Hall-effect, using samples supplied by G. Dorda and M. Pepper. In the same year they published a paper [vKDP80] in which the Quantum Hall-effect was presented to the world. The Quantum Hall-effect(QHE) is under certain conditions found in effectively two dimensional systems of electrons subjected to a strong magnetic field. In these systems the conductivity tensor takes the following off diagonal form

$$\sigma = \begin{pmatrix} 0 & -ie^2/h \\ ie^2/h & 0 \end{pmatrix} \tag{1.1}$$

Here h is Planck's constant, e is the electroncharge and i is a small integer.
In 1982 D. Tsui, H. Störmer and A. Gossard [TSG82] found experimentaly that in some very pure samples at very low temperature i could take on specific rational fractional values f as well. The fractions $f = p/q$ have odd denominators. The phenomena found by Tsui, Störmer and Gossard is called the fractional or anomalous quantum Hall-effect. The fractional quantum Hall-effect could not be explained by the theory for the integral quantum Hall-effect and another mechanism was needed. Now it is believed that the IQHE is stabilized by impurities in the sample where the FQHE is stabilized by quantization of interelectronic spacing and demands pure samples with no or only a few impurities.
Both quantum Hall-effects are observed in inversion layers, an interface between a semiconductor and an insulator or two semiconductors. The electrons live in a potential well created by a electrical field perpendicular to the interface, the motion perpendicular to the interface is thereby quantized and can be frozen

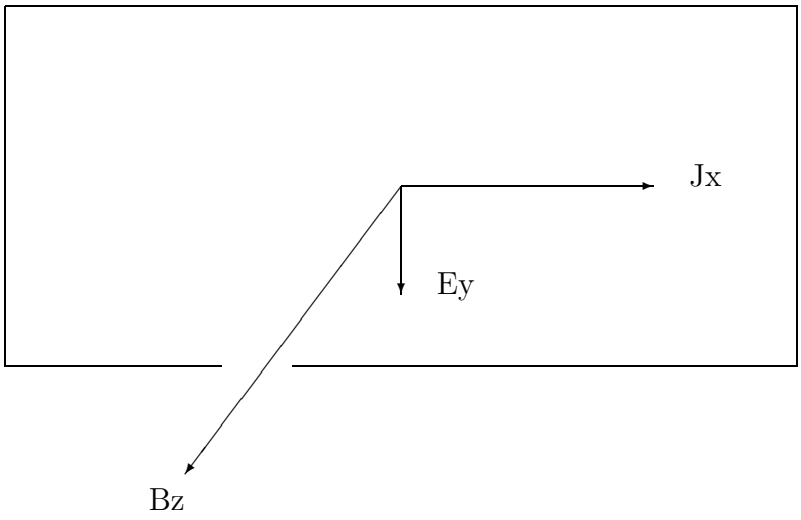


Figure 1.1: Hall Effect

out. The result is a two dimensional system of electrons. This electrical field can be external as in MOSFETS or internal as in heterostructures. The band structure around the interface in a heterostructure has for example the form given in figure [1.1] for an $Al_xGa_{1-x}As - GaAs$ heterostructure.
The $Al_xGa_{1-x}As$ layer is doped away from the interface so that it has electrons in the conduction band, these electrons move to the $GaAs$ layer to fill holes in the top of the valence band but most of them wil end up in states near the bottom of the conduction band. The electrons are attracted to the interface by the positive charge left on the donor impurities. The energy bands are thereby bended near the interface and thus the impurities give rise to the electric field in the system.
What is so remarkable about the quantum Hall-effect? If we look at picture [1.1] and consider the ordinary Hall-effect where the electrons live in three dimensions we would expect the following behaviour

$$\vec{E} = \rho\vec{j}, \rho = \begin{pmatrix} \rho_{xx} & -\rho_H \\ \rho_H & \rho_{yy} \end{pmatrix} \tag{1.2}$$

where $\rho_{xx} = R_L\frac{l_y}{l_x}$,$\rho_{yy} = R_L\frac{l_x}{l_y}$, l_x and l_y are the length and width of the system and R_L is the longitudinal resistance. ρ_H is in the ordinary Hall-effect given by

$$\rho_H = \frac{h}{e^2}\nu^{-1} \tag{1.3}$$

where ν is a dimensionless quantity called the filling factor

$$\nu = \frac{1}{B_c}(n\frac{h}{e}) \tag{1.4}$$

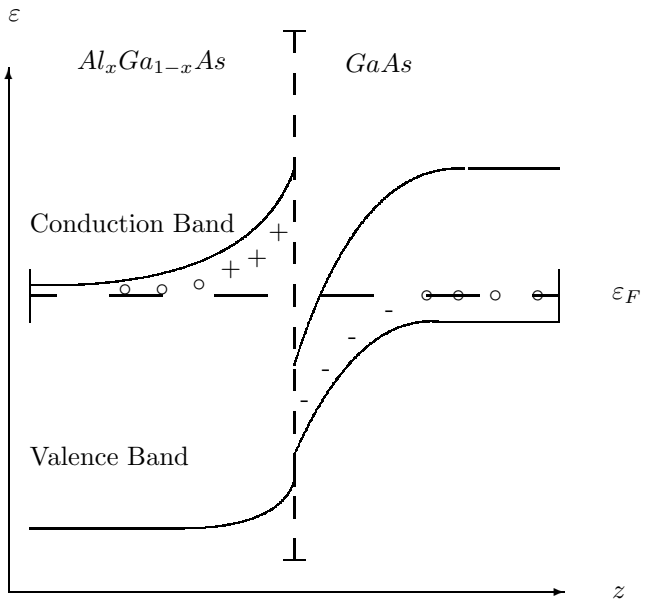


Figure 1.2: Energy levels in heterostructure

here e and h have their usual meaning, n is the conduction electron density and B_c is the magnetic field strength perpendicular to the two dimensional system. What we see is a Hall-conductivity ρ_H^{-1} which is linear in the filling factor ν. The magnetic field B_c can in experiment be manipulated and the linear behaviour can be tested experimentally. These experiments done in very pure heterojunctions at very low temperature show a remarkable behaviour, see figure [1.1]. The information in this picture from [Cha87] can be summarized

- $\sigma_H \equiv \nu \frac{h}{e^2}$ has plateaux at rational values
- When (ν, σ_H) belongs to a plateaux the longitudinal resistance R_L vanishes almost complete.

Furthermore it has been found that there exist fractionally charged excitations if (ν, σ_H) belongs to plateau at an non-integer value.

1.2 Landau levels,Localized and Extended states

The integer quantum Hall-effect can be explained by considering a system of non-interacting electrons confined to two dimensional space. Later we need the coulomb interaction to explain the fractional quantum Hall-effect. But for the

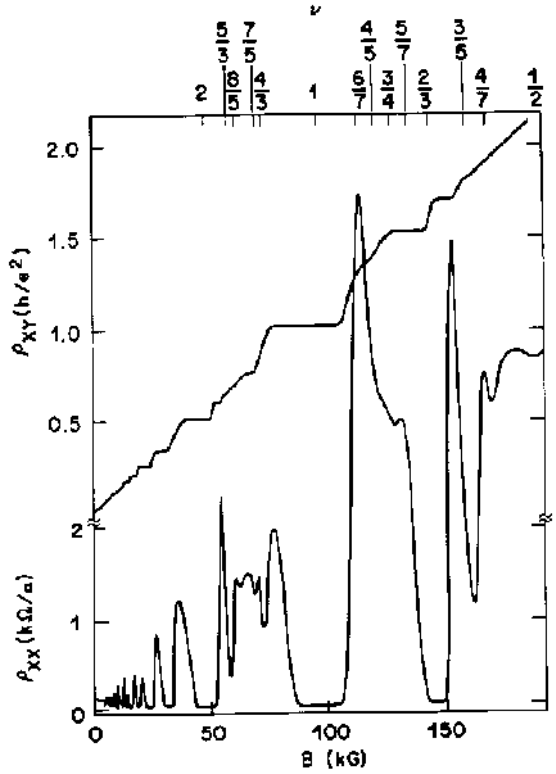


Figure 1.3: Hall-resistance as a function of the filling factor

moment we limit the analysis and look at an electron in a uniform magnetic field, we follow here the approach outlined in M. Stone's book [Sto92] in the first chapter . We choose the Landau-gauge

$$A_x = -By, A_y = 0 \tag{1.5}$$

and obtain the following one electron Schrödinger equation:

$$\left(-\frac{1}{2m}\partial_y^2 - \frac{1}{2m}(\partial_x - ieBy)^2\right)\psi(x,y) = E\psi(x,y) \tag{1.6}$$

This equation is translationaly invariant in the x direction, which suggests solutions of the following form

$$\psi(x,y) = e^{ikx}\phi(y) \tag{1.7}$$

and substituting this into the Schrödinger equation yields

$$\left(-\frac{1}{2m}\partial_y^2 + \frac{1}{2m}(k - eBy)^2\right)\phi(y) = E\phi(y) \tag{1.8}$$

The preceding equation is the equation for an shifted harmonic oscillator, and therefore it has (up to normalization) the following solutions for $\phi(y)$

$$\phi_{nk}(y) = \frac{1}{\sqrt{l2^n n!\sqrt{\pi}}} H_n(y/l - lk) exp[-(y - l^2k)^2/2l^2] \tag{1.9}$$

where H_n is an Hermite polynomial and l is the magnetic length $\sqrt{\frac{1}{eB}}$. The corresponding energy levels are independent of k

$$E_{nk} \quad = \quad \omega_c(n + \frac{1}{2}) \tag{1.10}$$

where $\omega_c = \frac{eB}{m_e c}$. The energy levels are completely determined by n the remaining freedom in choosing k leads to a degeneracy of the energy levels. If the system is confined in the y-direction with a width W the number of states in one full Landau-level can be calculated. The wave function is centered at $y = l^2k$, if we demand the center to be in the region between 0 and W we are able to put an constraint on the k-values

$$0 \leq k \leq W/l^2 \tag{1.11}$$

If we now impose the periodic boundary condition $\psi(x, y) = \psi(x + L, y)$ and using $k = 2\pi p/L$, we obtain the number of states in one level

$$N = LW/2\pi l^2 \tag{1.12}$$

We are now able to calculate the number of states per unit area of a full Landau-level

$$n_B = \frac{1}{2\pi l^2} = \frac{eB}{2\pi} \tag{1.13}$$

If we now modify the Hamiltonian to include an external electric field, we can calculate new wave functions. The Schrödinger equation becomes

$$\left(-\frac{1}{2m}\partial_y^2 - \frac{1}{2m}(\partial_x - ieBy)^2\right)\psi - \frac{eVy}{L_y}\psi = E\psi \tag{1.14}$$

And following the same procedure as before we obtain

$$\left(-\frac{1}{2m}\partial_y^2 - \frac{1}{2m}(k - ieBy - \frac{meV}{L_y})^2\right)\phi(y) = E\phi - \left(\frac{kV}{BL_y} - \frac{(mV)^2}{2mB^2L_y^2}\right)\phi \tag{1.15}$$

with the following solutions

$$\phi_{nk}(y) = \frac{1}{\sqrt{l2^n n!\sqrt{\pi}L_x}} H_n(y/l - lk - l^3 emV/L_y) exp^{-(y - l^2k - l^4 emV/L_y)^2/2l^2} \tag{1.16}$$

Now the current-density contribution of one Landau-level can be calculated

$$\begin{aligned} J_x(y) &= \frac{1}{2mi}(\psi^\dagger(\partial_x\psi) - (\partial_x\psi^\dagger)\psi + 2ieA_x\psi^\dagger\psi) \\ &= \frac{1}{l2^n n!\sqrt{\pi}L_x}\frac{l^3 emV/L_y - Y}{lm}H_n(Y)H_n(Y)e^{-eB(Y)^2} \end{aligned} \tag{1.17}$$

where we introduced $Y = y/l - lk - l^3 emV/L_y$. The integral over y will yield the current in the x-direction. First we separate the integral in a part which by inspection is clearly zero and a part that is nothing more as the innerproduct on the Hermite-polynomials multiplied by some constant.

$$\begin{aligned} I_\psi &= e\int \frac{1}{l2^n n!\sqrt{\pi}L_x}\frac{-Y}{lm}H_n(Y)H_n(Y)e^{-eB(Y)^2}dy \\ &\quad + e\int \frac{1}{l2^n n!\sqrt{\pi}L_x}\frac{l^2 eV}{L_y}H_n(Y)H_n(Y)e^{-eB(Y)^2}dy \\ &= \frac{eV}{BL_yL_x} \end{aligned} \tag{1.18}$$

The dependence on the Landau-level is lost in the calculation of the current carried by one state. Now we can calculate the current carried by one Landau level by multiplying by the electron density in one level, if we simultaneously sum over all the occupied Landau levels we obtain

$$\begin{aligned} I_{total} &= \sum_{Landau-level=1}^{n} n_B L_x L_y I_\psi \\ &= n\cdot\frac{eBL_xL_y}{2\pi}\cdot\frac{eV}{BL_xL_y} = \frac{ne^2V}{2\pi} \end{aligned} \tag{1.19}$$

In this formula we recognize the integer steps in the conductivity measured by K. von Klitzing [vKDP80].
We saw that the localization properties of the wave-functions depend on the gauge chosen, this represents a freedom in the choice of wave-functions describing the system. We could also chose the wave-functions to be localized but then we would find they would overlap thereby making a current possible. However impurities in the sample can give rise to states localized at the impurity, the role of these impurities will be discussed in the following section.

1.3 Hall-plateaux

If we look at the behaviour of the resistance versus the electron-density, where the electron-density is normalized to a multiple of the Landau-level electron-density we would expect the behaviour sketched in the preceding sections only for values of the electron density equal to an integer multiple of the Landau-level degeneracy per unit area. Experiments show however that the same behaviour

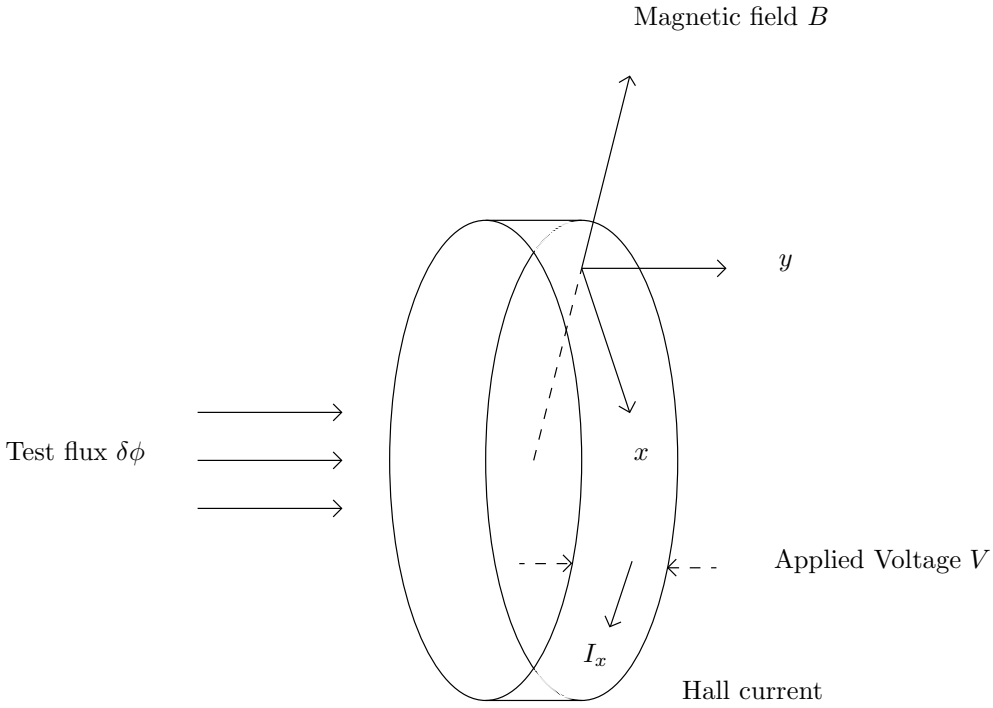


Figure 1.4: Special geometry for gauge argument

occurs in a neighbourhood of these integer values, and that the so called Hall-plateaux arise. The reasons for this behaviour can be explained by considering the influence of impurities on the behaviour of a system of non-interacting electrons in the presence of a gap in the spectrum of the Hamiltonian. There are several ways to explain this behaviour. We will present the gauge argument originally proposed by Laughlin [Lau81]
Laughlin considered a special geometry, see figure [1.3], a metalstrip bent into a loop of circumference L and pierced everywhere by a magnetic field B normal to its surface. In this geometry the potential drop V from edge to edge can be related to the current carried around the loop. This current can be found by functional differentiating the Hamiltonian with respect to the vectorpotential. The change of the vector potential is accompanied by a elektromotive force around the loop.

$$I_x = \frac{\delta H}{\delta A_x} = e\frac{< -\partial_x\psi|\psi > + < \psi|\partial_x\psi > + < \psi| - 2ieA_x|\psi >}{2mi} \quad (1.20)$$

or short

$$I_x = \frac{1}{L}\frac{\partial U}{\partial A_x} = \frac{\partial U}{\partial \phi_y} \quad (1.21)$$

Where ϕ_y is the total flux through the loop, U is the energy expectation value

and x the coordinate around the loop. If all the states are localized then the only effect of A is to multiply each state by e^{ieA_x}. In this case the current and the energy change are zero. If a state is extended there might be a current around the loop, however after a gauge transform the spectrum of the Hamiltonian should return to its original position. This gives us the restriction

$$A = \frac{2\pi j}{eL} \tag{1.22}$$

where j is a positive integer. In the case of non-interacting electrons an increase of the vector potential will force the filled states toward one edge of the ribbon. Because in the preceding section we adopted periodic boundary conditions, we can study the influence of the vector-potential within the same picture if we identify the period with the circumference of the loop. We are now able to read off the change in position of a state after a change in the vector-potential

$$\begin{aligned} \vec{A} &= -By\hat{x} \rightarrow \vec{A} = (-By - \Delta A)\hat{x} \\ &\Rightarrow \\ y_{center} \rightarrow y_{center} &= \left(kl^2 + l^4 emV/L_y + \Delta A/B\right) \end{aligned} \tag{1.23}$$

The energy of the state ϕ_{nk} is now given by

$$E_{n,k} = (n + \frac{1}{2})\omega_c + eVy_{center} + \frac{1}{2}m(V/L_yB)^2 \tag{1.24}$$

Now we can evaluate the current if we make the substitution

$$\frac{\partial U}{\partial \phi} \rightarrow \frac{\Delta U}{\Delta \phi} \tag{1.25}$$

with $\Delta\phi = 2\pi/e$ a flux quantum. Since by gauge invariance , adding one flux quantum maps the system back into itself, the energy increase is due to a net transfer of n electrons from one edge to the other, where n is the number of occupied Landau-levels . And we find

$$I_x = \frac{neV_y}{\Delta\phi} = \frac{ne^2V_y}{2\pi} \tag{1.26}$$

We now consider the influence of impurities on the system. There is a gap still, although the states are influenced by perturbations. As in the ideal case, gauge invariance is an exact symmetry, therefore addition of flux can only lead to excitation or deexcitation of the original system. Adiabatic change of the many-body Hamiltonian cannot give rise to excitations above the gap and therefore the only excitation possible is a transfer of electrons from one edge of the ribbon to the other . Although the number of electrons participating in the transfer need not be the ideal number, n times the degeneracy of one Landau-level, the conductivity remains unchanged.

Chapter 2

Fractional Quantum Hall-Effect

The fractional quantum Hall effect (FQHE) was discovered in 1982 by Tsui, Störmer and Gossard. Experimentally the FQHE is very similar to the IQHE, Hall plateaus in ρ_{xy} are accompanied by a (partially) vanishing of the longitudinal resistivity ρ_{xx}, there is however one important difference the effect occurs at fractional fillings. In 1983 Laughlin [Lau83] came up with an explanation for the simple filling fractions $\frac{1}{2k+1}$, in the same year Haldane [Hal83] generalized his approach to other filling fractions. Haldane found that every simple fraction was at the root of a hierarchy of states describing non-simple filling fractions like for example $\nu = \frac{2}{3}, \frac{3}{5}$. The hierarchy described by Haldane arises because excitations of the FQHE-states at a simple filling fraction can again form a FQHE-state and the excitations of this state can also form an FQHE-state etc. It was Halperin [Hal84] who realized that the excitations of the groundstate should satisfy fractional statistics, and he was able to reconstruct Haldane's Hierarchy assuming fractional statistics for the excitations.

To explain the simple filling fractions Laughlin introduced the concept of incompressibility this concept also plays a key role in the explanation of the occuring non-simple filling fractions. In order to translate this concept to a gauge theory Fröhlich und Studer [FS92] cast it in an other form. Our interest is exactly the connection between Laughlin's Incompressible Quantum Liquid and the gauge theory constructed by Fröhlich und Studer. In order to find fractional charged excitations in Laughlin's formulation the groundstate should be non-degenerate, later we will show that this demand might be equivalent to the incompressibility concept used by Fröhlich und Studer.

Now the original approach of Laughlin is sketched and its generalization by Haldane to non-simple filling fractions, in this generalization uniqueness of the groundstate is needed in order to construct fractional charged particles. These fractional charged particles satisfy fractional or braid statistics.

2.1 Laughlin Approach

To study the FQHE Laughlin first considered two dimensional systems containing only two or three electrons. He was able to extract from these systems the essential features of the FQHE, these features where generalized to a many-electron system. We follow the same path using the results obtained by Laughlin in *numerical* simulations.
We start from an idealized Hamiltonian

$$H = \sum_{j=1}^{n} \left[\frac{1}{2m_e} \left(\frac{\hbar}{i} \nabla_j + \frac{e}{c} A_j \right)^2 + V(r_j) \right] + \frac{1}{2} \sum_{j \neq k} \frac{e^2}{|r_j - r_k|} \tag{2.1}$$

where A_j is the vector potential for the symmetric gauge $A = \frac{B}{2}(y\hat{x} - x\hat{y})$ evaluated at the position of the j^{th}electron, and $V(r)$ is the potential generated by a neutralizing background. The spin-degrees of freedom are assumed to be frozen out by the magnetic field. If we evaluate this Hamiltonian with $V(r_j)$ independent of position then for one-electron we find highly degenerate states

$$\psi_{m,n}(x,y) = e^{\frac{1}{4}(x^2+y^2)} (\partial_x + i\partial_y)^m (\partial_x - i\partial_y)^n e^{-\frac{1}{2}(x^2+y^2)} \tag{2.2}$$

with the magnetic length $l = (\frac{\hbar c}{eB})^{\frac{1}{2}}$ set to unity. These eigenstates have energy values that depend on n only

$$E_{n,m} = (n + \frac{1}{2})\hbar\omega_c \tag{2.3}$$

where $\omega_c = \frac{eB}{m_e c}$ is the cyclotron frequency. Laughlin incompressibility can be understood looking at two or three electron systems. We will describe both systems starting with the two electron system. We have to formulate a few conditions:

- The Coulomb potential energy is small compared to the cyclotron energy, as a consequence the two electron wave function can be constructed from single electron wave functions in the lowest Landau level ($n = 0$)
- The two electron wave function is an eigenstate of internal angular momentum as a consequence of azimuthal symmetry of the Hamiltonian
- The two electron wave-function must be totally anti-symmetric in order to satisfy the Pauli-principle

The only two-electron wave functions satisfying these three conditions are up to center of mass motion

$$\psi(z) = (z_1 - z_2)^{2k+1} e^{-\frac{1}{4}(|z_1|^2 + |z_2|^2)} \tag{2.4}$$

where $z_j = x_j + iy_j$ is the transformation of the coordinates (x_j, y_j) of the j^{th} electron into a complex number and $2k + 1$ is an odd number. Using equation [2.4] one finds

$$\langle |z_1 - z_2|^2 \rangle \simeq 8k + 8 \tag{2.5}$$

where equality only holds in the limit that the electron-electron interaction is zero. Interelectronic spacing is now quantized because the eigenstates of the two-electron Hamiltonian do not depend on the electron-electron interaction if the interaction is sufficiently small. This property of the two electron system is thought to reflect incompressibility of Fractional Quantum Hall-systems.
Looking at three electron systems Laughlin found that the area occupied by three electrons becomes quantized. The three electron Hamiltonian can be rewritten using complex center of mass coordinates, which can be defined in the following way

$$\begin{aligned} \bar{z} &= \frac{z_1+z_2+z_3}{3} \\ z_a &= \sqrt{\frac{2}{3}}\left[\left(\frac{z_1+z_2}{2}\right)-z_3\right] \\ z_b &= \frac{1}{\sqrt{2}}(z_1-z_2) \end{aligned} \tag{2.6}$$

The Hamiltonian is now given by

$$\begin{aligned} H &= \frac{1}{2m_e}\left(\frac{\hbar}{i}\nabla_a+\frac{e}{c}A_a\right)^2+\frac{1}{2m_e}\left(\frac{\hbar}{i}\nabla_b+\frac{e}{c}A_b\right)^2 \\ &+ \frac{e^2}{\sqrt{2}}\left[\frac{1}{|z_b|}+\frac{1}{|\frac{1}{2}z_b+\frac{\sqrt{3}}{2}z_a|}+\frac{1}{|\frac{1}{2}z_b-\frac{\sqrt{3}}{2}z_a|}\right] \end{aligned} \tag{2.7}$$

where the center of mass motion has been removed. The Pauli principle is satisfied if the wave is odd in z_b and symmetric under rotation by ± 120 deg in the $a-b$ plane. A requirement that is satisfied when using the orthonormal basis functions

$$\begin{aligned} |m,n\rangle &= \frac{1}{\sqrt{2^{6m+4n+1}(3m+n)!n!\pi^2}} \\ &\cdot \left[\frac{(z_a+iz_b)^{3m}-(z_a-iz_b)^{3m}}{2i}\right]\cdot(z_a^2+z_b^2)^n e^{-\frac{1}{4}(|z_a|^2+|z_b|^2)} \end{aligned} \tag{2.8}$$

where $l=(\frac{\hbar c}{eB})^{\frac{1}{2}}$ is set to one. The lowest Landau level is spanned by these eigenstates of angular momentum with eigenvalues $M=2n+3m$. These states were evaluated by Laughlin numerically and they turned out to be good approximations to the correct energy eigenstates.
Now we can proceed to evaluate the "area" occupied by a three electron system which will provide us with more intuition of the incompressibility concept as introduced by Laughlin. What will become apparent is the remarkable stability of the three electron function against changes in the background potential. The background potential chosen is

$$\begin{aligned} V &= \frac{\alpha}{2}(|z_1|^2+|z_2|^2+|z_3|^2) \\ &= \frac{3}{2}\alpha|\bar{z}|^2+\frac{\alpha}{2}(|z_a|^2+|z_b|^2) \end{aligned} \tag{2.9}$$

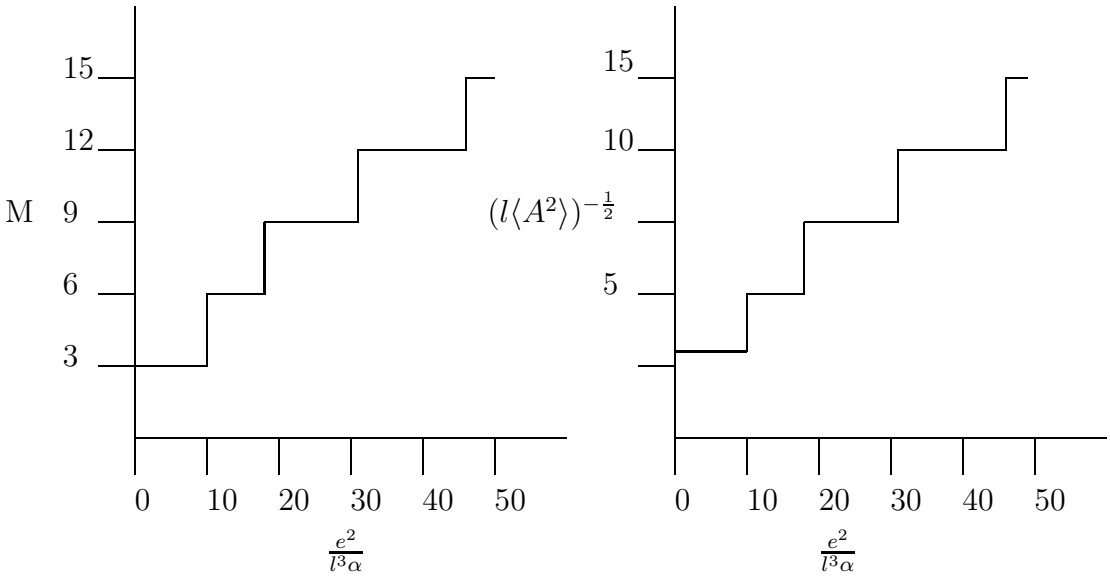


Figure 2.1: Area occupied by a three electron system as a function of the background potential

and is added to the Hamiltonian to hold the electrons together. This potential satisfies

$$\left\langle m,n|\frac{\alpha}{2}(|z_a|^2+|z_b|^2)|m',n'\right\rangle = \delta_{m,m'}\delta_{n,n'}(M+2)\alpha \tag{2.10}$$

This potential gives only a small energy contribution to states with a small angular momentum and makes these states energetically more favourable compared to high angular momentum eigenstates and thus it tends to keep the electrons together. This potential however has no influence on the shape of the eigenfunctions. The positions of the electrons span a triangle, the area of this triangle is given by

$$A = \frac{\sqrt{3}}{4i}(z_a z_b^* - z_b z_a^*) \tag{2.11}$$

This operator has the following matrix elements

$$\begin{aligned} < m,n|A|m',n' > &= 0 \\ < m,n|A^2|m',n' > &= \delta_{m,m'}\delta_{n,n'}\frac{3}{4}[(3m)^2+(M+2)] \end{aligned} \tag{2.12}$$

In figure [2.1] the results of Laughlin's calculations on the angular momentum and the expectation value of the area-operator A^2 as functions of α the parameter that defines the strength of the background potential are shown. The states with $n = 0$ have the lowest energy eigenvalues independent of α.Therefore the angular momentum shows discontinuous steps of three as α is decreased.

Variational Ground State

Laughlin generalized the intuition obtained in the preceding section to obtain a variational ground-state. As a starting point he took the variational ground state of liquid ^{3}He which is described by a single Slater determinant multiplied by a factor

$$\prod_{j<k}^{N} f(z_j - z_k) \tag{2.13}$$

where f is a function which goes to zero as its argument goes to zero, the product is called a Jastrow product. The Slater determinant is removed because the magnetic field effectively turns of the kinetic energy, at the same time the Jastrow product has to made antisymmetric in order to preserve Fermi statistics. Furthermore Laughlin factored out an exponential which is possible because it is equivalent to neglecting center-of-mass motion

$$\Psi(z_1, \ldots, z_N) = \prod_{j<k}^{N} f(z_j - z_k) e^{-\frac{1}{4}\sum_l^N |z_l|^2} \tag{2.14}$$

Now f can be varied until the expectation value of the energy becomes minimized, the intuition developed before can be used to remove variational freedom and leads to a unique ground state provided the following construction gives all possible eigenstates. The constraints now used are

- The wave function can be constructed from single electron wave functions in the lowest Landau level, and thus $f(z)$ is an analytic function of its argument z
- The wave function is totally antisymmetric, thus $f(z)$ must be odd
- The wave function is an eigenstate of angular momentum, conservation of angular momentum implies that

  $$\prod_{j<k}^{N} f(z_j - z_k) \tag{2.15}$$

  must be polynomial in the electron positions $z_1 \ldots z_N$ of degree M, where M is the total angular momentum.

Only $f(z) = z^m$, with m odd satisfies all these three constraints. Laughlin compared these wave functions with the exact numerically obtained wave functions for three interelectronic potentials and found them to be in good agreement. The ground state is to be found in a discrete set of wave functions characterized by an odd integer m

$$\begin{aligned} |m\rangle &= \Psi(z_1, \ldots, z_N) \\ &= \prod_{j<k}^{N} (z_j - z_k)^m e^{-\frac{1}{4}\sum_l^N |z_l|^2} \end{aligned} \tag{2.16}$$

Since $|m\rangle$ is an eigenstate of angular momentum with eigenvalue

$$M = \frac{N(N-1)}{2}m \tag{2.17}$$

mixtures of eigenstates are not allowed. The lowest energy state can be found by interpreting the square of the wave function as a classical probability function

$$|\Psi(z_1, \ldots, z_N)|^2 = e^{-\beta\phi(z_1,\ldots,z_N)} \tag{2.18}$$

The arbitrary fictitious temperature $\frac{1}{\beta}$ can be set equal to m and the effective classical potential becomes

$$\phi(z_1, \ldots, z_N) = -2m^2 \sum_{j<k} \ln|z_j - z_k| + \frac{m}{2}\sum_{l}^{N} |z_l|^2 \tag{2.19}$$

This is the potential of of a two-dimensional one-component plasma. This plasma consists of particles with "charge" m repelling one another via the natural "Coulomb" interaction in two dimensions $\sum_{j<k} ln|z_j - z_k|$ and attracted to the origin due to the same "Coulomb" interaction by a uniform neutralizing background of "charge-density"

$$\rho_{background} = \nabla^2 \frac{1}{2}\sum_{k}^{N} |z_k|^2 = \frac{1}{2\pi l^2} \tag{2.20}$$

A plasma strives for electric neutrality everywhere and therefore the electron density of the ground state

$$\rho_e(z_1) = \frac{N \int \ldots \int |\Psi_m(z_1, \ldots, z_N)|^2 d^2z_2 \ldots d^2z_N}{\langle m|m\rangle} \tag{2.21}$$

must be equal to $\frac{1}{m}$ times the "charge density" of the equivalent one-component plasma and thus uniform and equal to

$$\rho_m = \frac{1}{2\pi m l^2} \tag{2.22}$$

The states $\langle m|$ have significantly different charge densities. The groundstate is the state most closely reproducing the neutralizing background density ρ_0 in a physical sample. That this groundstate is indeed unique is supported by the experimental observation of non-simple filling fractions. To explain these filling fractions we need fractional charged excitations of the groundstate and these we can only construct assuming the groundstate to be non-degenerate.

2.2 Fractional Charged Quasiparticle Excitations

We now discuss the statistics of excitations of the FQHE-groundstate. These excitations have fractional charge and condensations of these excitations will

lead to non-simple filling fractions for the FQHE-groundstate. Haldane used these excitations to create the hierarchy of FQHE-states which explains non-simple filling fractions, later on Halperin and Laughlin did the same but they used excitations satisfying respectively fractional and Fermi statistics instead of Bose statistics as assumed by Haldane . Because Laughlin's formalism provides us with physical insight in the nature of the excitations we will treat the excitations within his formalism. Later on we use Berry's phase to determine the correct statistics for the excitations and we will see that the excitations satisfy fractional or braid statistics.
If there is no groundstate degeneracy one can make the existence of fractionally charged quasi-particles plausible and in view of the experimental evidence for the existence of fractional charges we assume the groundstate to be non-degenerate. The construction of the excitations starts with analyzing the effect of piercing the exact groundstate at location z_0 with an infinitely thin solenoid and adiabatically inserting flux through this solenoid a flux quantum $\phi_0 = \frac{hc}{e}$. Under adiabatic flux addition the wave-function changes being an eigenstate of the changing Hamiltonian. After addition of one flux quantum the Hamiltonian returns up to a gauge transformation to its original form at zero flux. The solenoid can be gauged away leaving an exact excited eigenstate of the original Hamiltonian, where the absence of degeneracy prevents the groundstate to move to another groundstate during the process.
Under addition of flux ϕ we see a behaviour of the single electron wave functions very similar to the behaviour of these functions in the gauge argument used in the theory of the IQHE. Under addition of flux the original one electron wave functions in the lowest Landau level

$$\Psi_{k,0} = (re^{i\chi})^k e^{-\frac{1}{4}r^2} \tag{2.23}$$

where $re^{i\chi} = z$ change to

$$\Psi^{\phi}_{k,0} = r^{|k+\frac{\phi}{\phi_0}|} e^{ik\chi} e^{-\frac{1}{4}r^2} \tag{2.24}$$

where the absolute value prevents a change of the wave function to a negative k value. After addition of a whole flux quantum these functions are up to a gauge transformation equivalent to

$$\Psi^{\phi_0}_{k,0} = z^{k\pm 1} e^{-\frac{1}{4}|z|^2} \tag{2.25}$$

where the sign depends on the sign of the flux added. We see that the addition or removal of one flux quantum amounts to increase or decrease the angular momentum by one, however the $k = 0$ state evolves under removal of one flux quantum to the next Landau level

$$\Phi^{\phi_0}_{0,0} = z^* e^{-\frac{1}{4}|z|^2} \tag{2.26}$$

The other states transform into their neighbor. By Schrieffer's counting argument [Lau83] the effect will be that the average charge per state is transported

out or into a large box surrounding the origin . The average charge per state is equal to $\frac{1}{m}$ where $m\frac{N(N-1)}{2}$ is the angular momentum asociated with the groundstate. Thus we expect that locally we will see a change in the charge present by exactly $\frac{1}{m}e$.
Based on the behaviour far from the origin with the Coulomb interaction between electrons turned off Laughlin assigned the following wave functions to the quasihole and quasiparticle created by adding one flux quantum to the groundstate

$$\begin{aligned} S_{z_0}|m\rangle &= e^{-\frac{1}{4}\sum_l^N |z_l|^2} \prod_i^N (z_i - z_0) \prod_{j<k}^N (z_j - z_k)^m \\ S_{z_0}^\dagger|m\rangle &= e^{-\frac{1}{4}\sum_l^N |z_l|^2} \prod_i^N (2\frac{\partial}{\partial z_i} - z_0^*) \prod_{j<k}^N (z_j - z_k)^m \end{aligned} \tag{2.27}$$

where S_{z_0} and $S_{z_0}^\dagger$ are respectively the quasihole and quasiparticle creation operator, $\partial_{z^*} = \frac{\partial_x + i\partial_y}{2}$ and $\partial_z = \frac{\partial_x - i\partial_y}{2}$ are the partial derivatives in complex coordinates. The creation operators can also be given in the form of ladder operators

$$\begin{aligned} S_{z_0} &= \prod_i^N (a_i^\dagger - z_o) \\ S_{z_0}^\dagger &= \prod_i^N (a_i - z_o^*) \\ a_j &= \frac{z}{2} + 2\partial_{z^*} \\ a_j^\dagger &= \frac{z^*}{2} - 2\partial_z \end{aligned} \tag{2.28}$$

where it has to be understood that these operators only work on the polynomial part of the wavefunctions. In this situation one can see the exponential part as a measure on these polynomials. If we now as in the previous section interpret the square of the quasi-hole wave function as the probability distribution function in classical statistical mechanics and set $\beta = \frac{1}{m}$

$$\langle \Psi_m^h, \Psi_m^h \rangle = e^{\frac{-1}{m}\Phi} \tag{2.29}$$

then we obtain for the classical potential ϕ

$$\Phi = \sum_{j<k} -2m^2 \ln|z_i - z_j| + \sum_l \frac{m}{2}|z_l|^2 - \sum_l 2m \ln|z_i - z_0| \tag{2.30}$$

Thus we have again a one component plasma with particles of charge m however there is an extra term which corresponds to a charge 1 at z_0, because in a

equilibrium the one component plasma is neutral everywhere the plasma will respond by building up a screening cloud around this charge of equal magnitude and opposite sign. However the system does not respond to the introduction of a charge but to the introduction of a flux quantum, the original charge is thus a phantom charge and the excitation has a charge -1. However the electron had charge m and thus translated back to the original system the excitation has charge $e^* = -\frac{e}{m}$.

Fractional Statistics

We mentioned before that the quasi-hole has charge $-\frac{1}{m}$, we will no determine the charge of the excitation in a more rigoruous way using Berry's phase. D. Arovas, J.R. Schrieffer and F. Wilczek [ASW84] originally made this analysis and they showed that Berry's phase can be used to find the statistics of the quasi-holes. We give a short discussion of the ideas behind Berry's phase, more details can be found in [Sch68] and [SW89]. The last book is especially recommended because its gives account of the basic theory behind Berry's phase and of the application of Berry's phase to the Fractional quantum Hall-effect and other physics problems.

Berry's phase

Berry's phase is found when adiabatically changing the Hamiltonian and assuming that at any time t there is a set of orthonormal time-independent non-degenerate eigenfunctions $\phi_n(t)$ satisfying the time-independent Schrödinger equation

$$H(t)\phi_n(t) = E_n(t)\phi_n(t) \tag{2.31}$$

where the spectrum of energy eigenvalues of the Hamiltonian is a discrete set. If we now adiabatically change the Hamiltonian it is expected that a eigen-function $\phi_n(0)$ at $t = 0$ will change continuously to an eigen function $\phi_n(t')$ at a time $t = t'$. This is also expected for the energy eigenvalues $E_n(t)$.
The wave-functions described above can be composed to form a wave-function Ψ solving the time dependent Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H(t)\Psi \tag{2.32}$$

the composition takes the following form

$$\Psi = \sum_n a_n(t)\phi_n(t)e^{\frac{1}{i\hbar}\int_0^t E_n(t')dt'} \tag{2.33}$$

If we restrict the analysis to $H(t)$, $0 \leq t \leq t'$, which all have the same set of eigenfunctions with corresponding energies the only freedom left to change an eigen-function during the adiabatic change of the Hamiltonian is in its phase. This is for example the situation if we transport a quasi-hole excitation of a

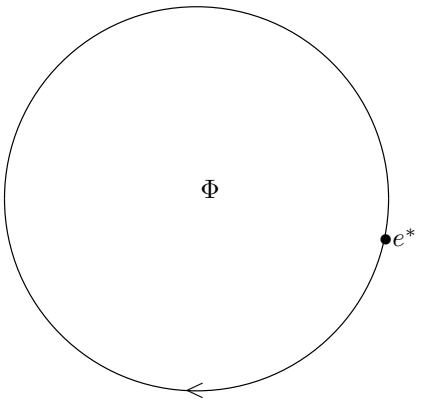


Figure 2.2: Aharonov-Bohm Effect

Laughlin groundstate. This phase change, Berry's phase, is given by the following expression and its derivation can be found in [Sch68] on page 289-291 and in a contribution of M.V. Berry to [SW89]

$$\frac{d}{dt}\gamma_{\text{Berry}}(t) = i\langle\phi(t)|\frac{d}{dt}|\phi(t)\rangle \tag{2.34}$$

An example of Berry's phase is found in the Aharonov-Bohm effect, if a wave function describing a particle (quasi-hole) with charge e^* is transported around a region of space containing a flux Φ, see figure [2.2], it changes its phase by

$$\frac{e^*}{\hbar c}\oint \vec{A}\cdot\vec{l} = 2\pi\frac{e*}{e}\frac{\Phi}{\Phi_0} \tag{2.35}$$

where Φ_0 denotes one flux quantum. For a quasi-hole transported around a loop $z_0(t')$ with $z_0(0) = z_0(t)$ we can use the quasi-hole wave-functions Ψ^h_m to find Berry's phase

$$\frac{d}{dt}\Psi^h_m(z_0(t)) = N_h\sum_{i=1}^{n}[\frac{d}{dt}\ln(z_i - z_0(t))]\Psi^h_m(z_0(t)) \tag{2.36}$$

If we plug this into equation [2.34] and use the following expression for the one electron density

$$\rho^{h,z_0}(z) = \langle\Psi^h_m(z_0)|\sum\delta(z_i - z)|\Psi^h_m(z_0)\rangle \tag{2.37}$$

we find the following relation for Berry's phase,

$$\frac{d}{dt}\gamma_{\text{Berry}} = i\int dxdy\rho^{h,z_0}(z)\frac{d}{dt}\ln(z - z_0(t)) \tag{2.38}$$

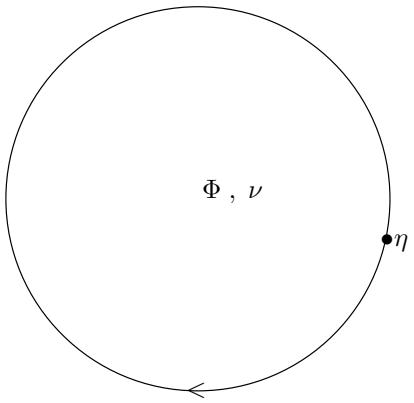


Figure 2.3: Berry's Phase for a quasi-particle

But we know the electron density of the groundstate of the fractional quantum Hall effect is equal to

$$\rho_0 = \frac{\nu B}{\Psi_0} \tag{2.39}$$

and we define

$$\rho^{h,z_o}(z) = \rho_0 + \delta\rho^{h,z_0}(z) \tag{2.40}$$

If we now integrate $\frac{d\gamma_{\text{Berry}}}{dt}$ clockwise around a loop L,see figure [2.3], the values of z inside the loop contribute $2\pi i$ to the integral where the values of z outside the loop give no contribution, furthermore the $\delta\rho^{h,z_0}(z)$ part gives no contribution to the integral due to spherical symmetry [SW89] p308. We find for γ_{Berry}

$$\begin{aligned} \gamma_{\text{Berry}}(L^{\text{clockwise}}) &= i\int_{z-\text{inside L}} dxdy\rho_0 2\pi i \\ -2\pi\langle n\rangle_L &= -2\pi\nu\frac{\Phi}{\Phi_0} \end{aligned} \tag{2.41}$$

where $\langle n\rangle_L$ denotes the expectation value the operator counting the electrons inside the loop and ν denotes the filling fraction and Φ is the flux inside the loop. Comparing equation [2.35] with equation [2.41] we find the quasi-hole charge

$$e^* = -\nu e \tag{2.42}$$

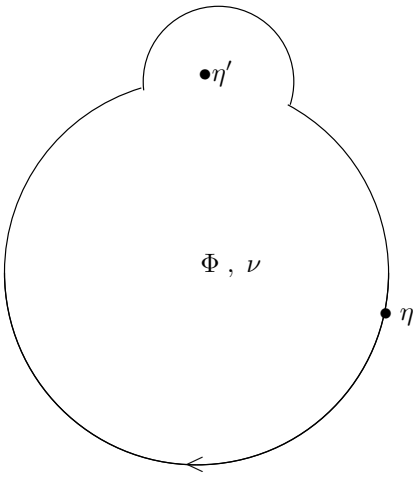


Figure 2.4: Taking one quasi-particle around the other

Quasi-hole Statistics

We can now use equation [2.41] to determine quasi-hole statistics. We move one quasi-hole into the loop and the other quasi-hole is transported around the loop as before. Moving one quasi-hole into the loop changes the expectation value $\langle n \rangle_L$ by an amount proportional to the quasi-hole charge $-\nu e$, that is Berry's phase for transporting a quasi-hole clockwise around the other along a loop enclosing flux Φ,see figure [2.4], becomes

$$\gamma_{\text{Berry}}(L^{\text{clockwise}}) = -2\pi\nu\frac{\Phi}{\Phi_0} + 2\pi\nu \tag{2.43}$$

The phase change asociated with interchanging two quasi-holes is half the additional phase change found when a quasi-hole is inside the loop, because interchanging particles is equivalent to transporting one particle around half a circle with the second particle in the centre and then translating the whole system back to the original position. However the sign of the phase change depends on whether the particles are interchanged clockwise or anticlockwise, if the quasi-hole is transported anticlockwise Berry's phase changes in the following way

$$\gamma_{\text{Berry}}(L^{\text{anti-clockwise}}) = -\gamma_{\text{Berry}}(L^{\text{clockwise}}) \tag{2.44}$$

putting everything together we find on interchanging two quasi-holes and leaving out the phase changes due to other particles present

$$\gamma_{\text{Berry}}(\text{Interchange Path}) = k\pi\nu,\ k \in \mathbb{Z} \tag{2.45}$$

where ν is in general non-integer in contrast to the case of Fermi or Bose statistics where the phase change corresponds with even or odd integer values of ν.
An elegant way of describing the paths followed by the quasi-holes during interchange is by braids. We will set up a correspondence between braids and phase changes by identifying operations on braids with phase changes. Thus obtaining an abelian representation of the braid group representing the fractional or braid statistics of the quasi-holes. The braid group ,B_n, on n infinitely long strands can be defined by its generators

$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} \tag{2.46}$$

for $i = 1, \ldots, n-2$ and

$$\tau_i\tau_j = \tau_j\tau_i \tag{2.47}$$

for $|i-j| \geq 2$, the inverse of τ_i is denoted by τ_i^{-1}, for all i and the identity element of B_n is denoted by 1. The center of B_n is generate by $(\tau_1, \tau_2, \ldots, \tau_{n-1})^n$. If we have given n-strands in $\mathbb{R}^3$, see figure [2.2], the generators τ_i act on these

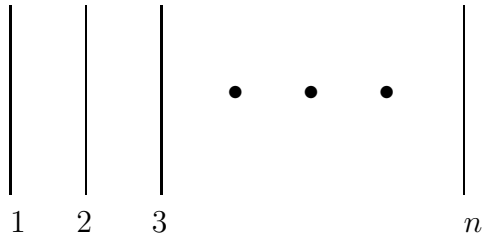


Figure 2.5: n-strands

strands by braiding the i^thstrand once, see figure [2.2]. The inverse of τ_i acts

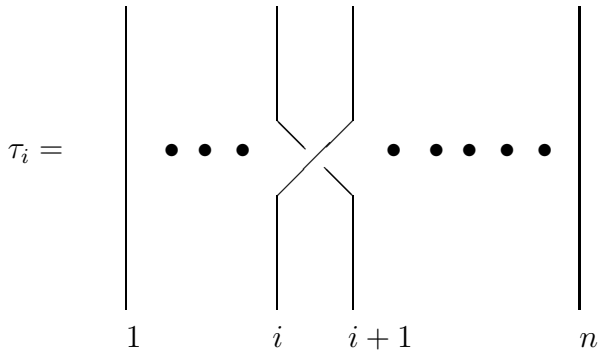


Figure 2.6: generator of the braid group

on the n-strands by braiding them in the opposite direction, see figure [2.2]. We can define a multiplication in B_n by sealing together the strands of the braid as in figure [2.2].

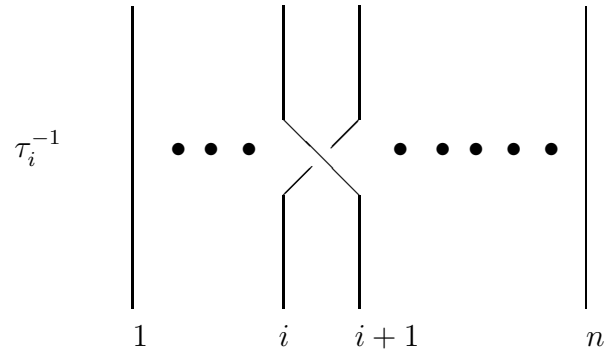


Figure 2.7: inverse of a generator

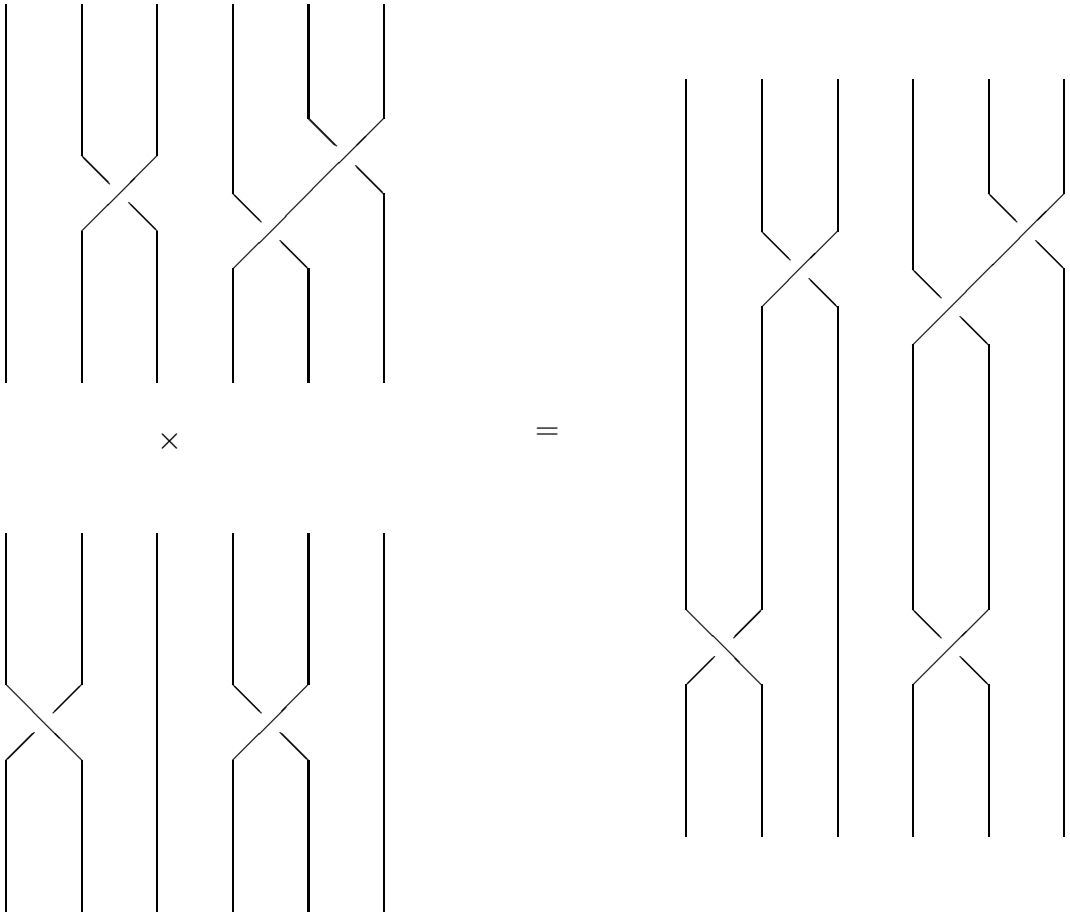


Figure 2.8: Product of two braids

If we identify τ_i with interchanging quasiparticle i with quasiparticle $i+1$, where the interchange is implemented by rotating $i+1$ through an angle π around i and then translating the quasi-particles back to the original positions but interchanged and if we identify τ_i^{-1} with the same interchange but with the rotation performed over an angle $-\pi$ then this identification defines a homomorphism from the braidgroup B_n to operations interchanging quasi-particles, where we denote operators asociated with τ_i by $\hat{\tau_i}$. We can see that the $\hat{\tau_i}$'s satisfy the equations [2.46] and [2.47] as the generators of the braid algebra did from the following relations

$$\begin{aligned} & \hat{\tau_i}\hat{\tau_{i+1}}\hat{\tau_i}\Psi_1\dots\Psi_i\Psi_{i+1}\Psi_{i+2}\dots\Psi_n \\ = \; & \hat{\tau_{i+1}}\hat{\tau_i}\hat{\tau_{i+1}}\Psi_1\dots\Psi_i\Psi_{i+1}\Psi_{i+2}\dots\Psi_n \\ = \; & e^{3\pi\nu}\Psi_1\dots\Psi_{i+2}\Psi_{i+1}\Psi_i\dots\Psi_n \end{aligned} \tag{2.48}$$

where $\Psi_i = \Psi(\vec{x}_i)$ and from

$$\hat{\tau_i}\hat{\tau_j} = \hat{\tau_j}\hat{\tau_i} \tag{2.49}$$

if $|i-j| \geq 2$ which holds because $\hat{\tau_j}$ and $\hat{\tau_i}$ both interchange quasi-holes but no quasi-hole is subjected to both interchange operators because of $|i-j| \geq 2$. The wave function should up to phase be invariant under permutation of the quasi-particles, but we have seen now that the phase depends on the way we interchange the quasi-particles, which is characteristic for fractional statistics.

2.3 Hierarchy of FQHE-states

We have seen how Laughlin obtained simple filling fractions $\frac{1}{n}$ with n an odd integer, Haldane generalized Laughlin's approach and was thereby able to predict the non-simple filling fractions. Haldane obtained his results by putting a 2-dimensional electron gas of N-electrons on a spherical surface of radius R, in a radial (monopole) magnetic field $B = \frac{\hbar S}{eR^2}$. Where $2S$ is the total magnetic flux through the surface and will turn out to be equal to an integer multiple of the unit of flux quantization $\Phi_0 = \frac{h}{e}$, a direct consequence of the commutation relations. In this setting homogeneous states with finite N can be constructed; furthermore the states are rotational invariant and in the thermodynamic limit R,N,and $S \to \infty$ it turns into translational invariance in the plane. The single particle Hamiltonian in this setting becomes

$$H = \frac{|\vec{\Lambda}|^2}{2m_e} \tag{2.50}$$

where $\vec{\Lambda}$ is the dynamical angular momentum

$$\vec{\Lambda} = \vec{r} \times [-i\hbar\vec{\nabla} + e\vec{A}(\vec{r})] \tag{2.51}$$

where $\vec{\nabla} \times \vec{A} = B\hat{\Omega}$, $\hat{\Omega} = \frac{\vec{r}}{R}$. By construction $\vec{\Lambda}$ has no component normal to the surface and has the following commutation relations

$$[\Lambda^\alpha, \Lambda^\beta] = i\hbar\epsilon^{\alpha\beta\gamma}(\Lambda^\gamma - \hbar S\Omega^\gamma) \tag{2.52}$$

The generator of rotations is given by

$$\vec{L} = \vec{\Lambda} + \hbar S\hat{\Omega} \tag{2.53}$$

and the following commutation relations hold

$$[\vec{L}^\alpha, \vec{X}^\beta] = i\hbar\epsilon^{\alpha\beta\gamma}\vec{X}^\gamma \tag{2.54}$$

where $\vec{X} = \vec{L}$,$\vec{\Omega}$, or $\vec{\Lambda}$. $\vec{L}$ has a component normal to the surface $\vec{L}\cdot\hat{\Omega} = \hat{\Omega}\cdot\vec{L} = \hbar S$. This algebra implies the spectrum $|\vec{L}|^2 = \hbar^2 l(l+1)$ where $l = S + n$ and $l = 0, 1, 2, ...$ and $2S$ is the radial component of $\vec{L}$ and thus an integer and thus n is integer; n denotes the Landau-level of a particular eigenstate of the Hamiltonian.

$$\begin{aligned} |\vec{\Lambda}|^2 &= |\vec{L}|^2 - \hbar^2 S^2 \\ &= \hbar^2[n(n+1) + (2n+1)S] \end{aligned} \tag{2.55}$$

$\hat{\Omega}$ can be expressed in spinor coordinates

$$\begin{aligned} u &= \cos(\frac{1}{2}\theta)e^{\frac{1}{2}i\phi} \\ v &= \sin(\frac{1}{2}\theta)e^{-\frac{1}{2}i\phi} \end{aligned} \tag{2.56}$$

in these coordinates

$$\hat{\Omega}(u, v) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) \tag{2.57}$$

The wave functions are described in the gauge $\vec{A} = \frac{\hbar S}{eR} \times \hat{\phi}\cot(\theta)$. In this gauge two singularities at the two poles exist, these singularities are a consequence of using one chart to describe the sphere and can be lifted by using two charts, see for example [Nak90](p14-p15). After looking at the one and two electron states on the sphere Haldane inspired by Laughlin's preceding work wrote down the following N-particle wave function

$$\begin{aligned} \Psi_N^{(m)} &= \prod_{i<j}(u_i v_j - u_j v_i)^m \\ S &= \frac{1}{2}m(N-1) \end{aligned} \tag{2.58}$$

this wave function has the advantage that it is translationally and rotational invariant, because it is an eigenfunction of $\vec{L}_{tot} = \sum_i \vec{L}_i$. The symmetry character of the function $\Psi_N^{(m)}$ depends on the exponent m if m is odd it obeys Fermi statistics if m is even it obeys Bose statistics. The choice for m is thereby for a system

of electrons restricted to odd values. The state with $S = (N; m) \equiv \frac{1}{2}m(N-1)$ describes an exact nondegenerate groundstate of the "projection-operator interaction potential", $H^{int}_{m,S}$, which satisfies

$$\Pi_S H^{int}_{m,S} \Pi_S = \sum_{i<j} [\sum_{J>2S-m} P_J(\vec{L}_i + \vec{L}_j)] \tag{2.59}$$

where the J's are eigenvalues of the angular momentum operators $\vec{L}_{ij} \equiv \vec{L}_i + \vec{L}_j$ and the $P_J(\vec{L}_i + \vec{L}_j)$ are projection operators onto states with $|\vec{L}_{ij}|^2 = \hbar^2 J(J+1)$ and Π_S is a projection operator onto states of the lowest Landau level. $\Psi^{(m)}_N$ is a good variational approximation for the groundstate of a system of particles with strong repulsion at close separations because the projection-operator interaction potential gives no contribution to the energy for $J_{ij} \leq 2S-m$, which corresponds to large separation of the electrons , and does give a positive contribution to the energy for $J_{ij} > 2S - m$ which corresponds to a small separation between the electrons.
The excitation operators are given by

$$A^\dagger_N(\alpha, \beta) = \prod_{i=1}^{N} (\beta u_i - \alpha v_i)$$
$$A_N(\alpha, \beta) = \prod_{i=1}^{N} (\beta^* \partial_{u_i} - \alpha^* \partial v_i) \tag{2.60}$$

the first operator represents a quasihole of charge $-\frac{e}{m}$ the second a quasiparticle of charge$\frac{e}{m}$. The operators respectively increase or decrease the flux quantum number S by $\frac{1}{2}$. A state with N^{ex}_p quasi particles and N^{ex}_h quasi holes has $S = S(N, m) + \frac{1}{2}(N^{ex}_h - N^{ex}_p)$. Following Laughlin Haldane assumed that for some m the groundstate wave function of a two dimensional electron system is well represented by the $\Psi^{(m)}_N$ and there exist a gap in the excitation spectrum, the lowest excitations being quasihole qausiparticle pairs. If in this situation the field strength is modified so that $S = S(N, m) + \frac{1}{2}N^{ex}$; the low energy states can be considered as deriving from the state $\Psi^{(m)}_N$ with an imbalance between quasiparticle and quasihole excitations, $N^{ex} \equiv N^{ex}_h - N^{ex}_p \neq 0$. Haldane argued that if the interaction between the quasi-particles or quasi-holes is small compared to the energy gap that then one can construct a collective groundstate of the excitation fluid analogous to the construction of the groundstate of the electron fluid. That is the excitations are considered to behave like charged particles with charge $\pm\frac{e}{m}$ living in the magnetic field together with the electrons but with behaviour independent of the electrons.
We now first present the construction of the Hierarchy in a simple intuitive way and after that we return to Haldane's treatment. The degeneracy of a Landau-level of excitations is $\frac{e^*}{e} = \pm\frac{1}{m}$ times the degeneracy of Landau-level of electrons. A filling $\frac{1}{p}$ of the Landau-level of excitations thus corresponds to the presence of $N^{ex} = \frac{1}{p}N$ excitations. A state describing the composite system

of electrons and excitations can be obtained from a state describing a simple filling fraction by adding or removing flux, creating one excitation corresponds exactly to adding or removing one quantum of flux. Every electron is asociated with m flux quanta thus we obtain for the total flux $2S = mN \pm \frac{N}{p}$ where the sign depends on whether the excitations are hole-like (+ sign) or particle like (- sign). Every flux quantum is asociated with one one-electron state and hence the filling fraction turns out to be

$$\nu = \frac{N}{2S} = \frac{1}{m \pm \frac{1}{p}} \tag{2.61}$$

this type of reasoning can be repeated yielding the Hierarchy.
We now return to Haldane's treatment. In the thermodynamic limit we have for every unit of magnetic flux $\Phi_m = m\Phi_0$ a one-excitation state with charge $\pm e/m$. To obtain the S asociated with the simple filling factor groundstate in units Φ_m we have to divide $\frac{1}{2}m(N-1)$ by m and we obtain $\frac{1}{2}(N-1)$. Assuming a filling $\frac{1}{p}$ for the one-excitation states we obtain

$$\begin{aligned} \frac{1}{2}(N-1) &= p|N^{ex}| \\ &\Leftrightarrow \\ \frac{1}{2}N &= S(|N^{ex}|;p) \end{aligned} \tag{2.62}$$

where p is now even (Bose statistics). Solving this equation leads to $|N^{ex}| = \frac{N}{p}+1$ and thus these states can occur at a total flux $2S$ with $S = S(N;m,\pm p) \equiv \frac{1}{2}m(N-1)\pm\frac{1}{2}(\frac{N}{p}+1)$ where N should be divisible by p. If this collective groundstate exists the chain of reasoning can be used again to construct a type $[|p_1|, p_2]$ collective groundstate and so on. We noted before that in the thermodynamic limit the degeneracy of the one-electron wave functions becomes equal to the total flux through the sphere, this enables us to calculate the filling fraction $\nu = \frac{N}{2S}$ in this limit.
The hierarchy obtained by repeatedly applying the above procedure can be summarized in the following equations describing the total flux, the number of excitations and the filling fraction

$$\begin{aligned} S(N;m,p_1,\ldots.P_n) &= S(N;m) + \frac{1}{2}|N^{ex}|\mathrm{sgn}(p_1) \\ \frac{1}{2}N &= S(|Nex|;|p_1|,p_2,\ldots,P_n) \end{aligned} \tag{2.63}$$

$$\nu(m,p_1,\ldots,P_n)^{-1} = m + \mathrm{sgn}(p_1)\nu(|p_1|,p_2,\ldots,p_n) \tag{2.64}$$

the above equation is solved by the continued fractions

$$[m,\alpha_1 p_1,\ldots,\alpha_n p_n] = \cfrac{1}{m + \cfrac{\alpha_1}{p_1 + \cfrac{\alpha_2}{\ddots \cfrac{}{+\frac{\alpha_n}{p_n}}}}} \tag{2.65}$$

where the $p_i := |p_i|, \alpha_i := \operatorname{sgn}(p_i)$. If ν is expresssed as a rational $\frac{P}{Q}$ where Q is odd then in the corresponding system the charge of an excitation becomes $e^* = \frac{e}{Q}$ and the Hall-resistance $\rho_{xy} = \frac{\Phi_0}{Pe^*} = \nu\frac{\Phi_0}{e}$.

Chapter 3

Effective Gauge Theory for Quantum Hall-Effects

The contents of this chapter are based on sections 2 and 3 of ref [FS92], in which J. Fröhlich and U.M. Studer study the Large scale behaviour of two dimensional systems of particles with spin and electromagnetic charge. In their paper they assume strong clusterproperties asserting these model the gap existing in the spectrum of the Hamiltonian. Such a system should furthermore satisfy $U(1) \times SU(2)$ gauge invariance. The central idea is that at low temperatures the energy gap in the system effectively removes the energy states above the gap thereby causing the large scale behaviour to be independent of the details of the interaction potential.

3.1 Effective Gauge Field Action in the Scaling Limit

In this section a covariant Hamiltonian, describing a system of particles with electromagnetic charge and spin, is defined. Using this Hamiltonian we define an action functional. Then a path integral is used in order to define the effective action. In the next section the effective action will be used to study the large scale behaviour of the system, there we leave out the $SU(2)$ part of the theory by setting the appropriate parameters to zero. Then we show that in the scaling-limit the $U(1)$-gauge-theory is a Chern-Simons Theory , we need Chern-Simons theory when we look at the duality proposition in chapter 5. In the paper we refered to J. Fröhlich und U.M. Studer proceed to examine the full $U(1) \times SU(2)$-theory and to construct a linear response theory leading among other things to the Hall-conductivity. From this analysis we only reproduce the interpretation of the effective action.
A system of electrons subjected to an electromagnetic field is described by second quantized , two component Pauli-spinor fields , Ψ and Ψ^*. The Pauli spinor

fields are operator valued sections of a complex vector bundle, Σ. The base space of Σ is $\mathbb{M}^3 = \mathbb{R} \times \Omega$ the $(2+1)$-dimensional space time of the electron system, where $\mathbb{R}$ is the time axis and Ω is surface of the sample. The fibre is $\mathbb{C}^2$ and the structure group $G = U(1) \times SU(2)$. Because the base space is simply connected the bundle is trivial, that is $\Sigma \simeq \mathbb{M}^3 \times G$, thus the bundle is not only a local produkt but also a global produkt of the base space with the fibre. Triviality of the bundle is equivalent to the existence of a section and thus the spinor-fields are well-defined.
Now we define canonical equal-time anti-commutation relations for the Pauli-spinor fields giving Fermi-statistics

$$\begin{aligned} \left[\psi^{\#}_{\alpha}(\vec{x}), \psi^{\#}_{\beta}(\vec{y})\right] &= 0,\ \vec{x}, \vec{y} \in \Omega \\ \left[\psi_{\alpha}(\vec{x}), \psi^{*}_{\beta}(\vec{y})\right] &= \frac{1}{\sqrt{g(\vec{x})}} \delta_{\alpha\beta} \delta(\vec{x} - \vec{y}) \end{aligned} \tag{3.1}$$

where $\psi^{\#} = \psi$ or ψ^* are two component spinor operators respectively the annihilation and the creation operator and the indices α, β refer to the spin up and spin down with respect to B_z. $g(\vec{x})^{-\frac{1}{2}} \delta(\vec{x} - \vec{y})$ is the Dirac δ-function in an arbitrary metric g_{kl}, on Ω, with $g(\vec{x})$ the determinant of g_{kl}, for the sake of simplicity we assume the metric to be independent of $\vec{x}$ and euclidean and hence $g(\vec{x}) = 1$.
On the bundle Σ we have the covariant derivatives

$$\begin{aligned} D^{\omega}_{\mu} &= \nabla_{\mu} + i a_{\mu}(x) + w_{\mu}(x) \\ &= \nabla_{\mu} + \frac{ie}{\hbar c} A_{\mu} + \frac{i\mu_e}{\hbar c} \vec{W}_{\mu}(x) \cdot \vec{S}, \end{aligned} \tag{3.2}$$

where ∇_μ is the Riemannian derivative, $x = (x_o, \vec{x}) \in \Sigma$, A_μ is the electromagnetic vector potential, $\vec{W}_\mu$ is a $SU(2)$ gauge potential and μ_e is the magnetic moment of the electron. a_μ and w_μ are the geometrical counterparts of the potentials also appearing in the connection one-form of the bundle Σ

$$\begin{aligned} \omega(x) &= (a, w)(x) \\ &= (a_{\mu}(x) dx^{\mu}, w_{\mu}(x) dx^{\mu}) \\ w_{\mu}(x) &= i \vec{w}_{\mu} \cdot \vec{\sigma} \end{aligned} \tag{3.3}$$

where a_μ and w_μ take their values in the Lie-algebra $\mathfrak{u}(1)$ respectively $\mathfrak{su}(2)$
The electromagnetic current density $j(x)$ and the spin current density $\vec{s}(x)$ are defined in the following way

$$\begin{aligned} j_0(x) &= \psi^*(x)\psi(x) \\ j_k(x) &= -\frac{i\hbar}{2mc}[(D_k\psi)^*(x)\psi(x) - \psi^*(x)(D_k\psi)(x)] \\ \vec{s}_0(x) &= \psi^*(x)\vec{\sigma}\psi(x) \\ \vec{s}_k(x) &= -\frac{i\hbar}{2mc}[(D_k\psi)^*(x)\vec{\sigma}\psi(x) - \psi^*(x)\vec{\sigma}(D_k\psi)(x)] \end{aligned} \tag{3.4}$$

The electromagnetic current obeys the following relation

$$\begin{aligned} d*j &\equiv d\mathcal{J} \\ d\mathcal{J} &= 0 \end{aligned} \tag{3.5}$$

where the $*$-operator is defined by taking the Hodge dual

$$\begin{aligned} \mathcal{J} &= \sum_{\mu<\nu} \mathcal{J}_{\mu\nu} dx_\mu \wedge dx_\nu \\ \mathcal{J}_{\mu\nu} &= |\eta(x)|^{1/2} \varepsilon_{\mu\nu\rho} \eta^{\rho\sigma}(x) j_\sigma(x) \end{aligned} \tag{3.6}$$

where $\eta_{\mu\nu}(x_0, \vec{x})$ is the metric in a (2+1)-dimensional Minkowski-space with $-g_{kl}(\vec{x})$ as its spatial part. In (2+1)-dimensions the Hodge *-dual $\mathcal{I}$ to a conserved current i can be written as the exterior derivative of a connection 1-form $\mathcal{A}$, that is

$$\begin{aligned} \mathcal{I} &= d\mathcal{A},\ \mathcal{I} = *i \\ d\mathcal{I} &= d^2\mathcal{A} = 0 \end{aligned} \tag{3.7}$$

If we take $i = j$ we obtain our previous statement $d\mathcal{J} = 0$, furthermore we see that the connection 1-form $\mathcal{A} + d\chi$ corresponds to the same current j as the original connection 1-form $\mathcal{A}$. Therefore $\mathcal{A}$ is an abelian gauge field and the effective theory of conserved current is an abelian gauge theory. In general the spin-current $\vec{s}$ is not conserved.
We are now ready to formulate a time-dependent Hamiltonian demanding general covariance

$$H(t) = H_0(A_\mu(x^0, .), \vec{W}_\mu(x^0, .), g_{kl}(x^0, .)) + H_I(t) \tag{3.8}$$

where

$$\begin{aligned} H_0(t) &= H_0(A_\mu(x^0, .), \vec{W}_\mu(x^0, .), g_{kl}(x^0, .)) \\ &= \frac{\hbar^2}{2m} \int_\Omega g^{kl}(x^0, \vec{x}) (D_k\psi)^*(\vec{x}) (D_l\psi)(\vec{x}) \sqrt{g(x^0, \vec{x})} d^2\vec{x} \\ &+ \int_\Omega (e j_0(\vec{x}) A_0(x^0, \vec{x}) + \mu_e \frac{\hbar}{2} \vec{s}_0(\vec{x}) \cdot \vec{W}_0(x^0, \vec{x})) \sqrt{g(x^0, \vec{x})} d^2\vec{x} \end{aligned} \tag{3.9}$$

and where m is the effective mass of the electron, and H_I describes electron-electron interactions and the interaction of electrons with the background. H_I does not not depend on A_μ and W_μ and is chosen to be gauge invariant. Now the dynamics of the spinor fields in the Heisenberg picture is given by the following equation

$$i\hbar \frac{\partial}{\partial x^0} \psi^\#(x^0, \vec{x}) = -[H(x^0), \psi^\#(x^0, \vec{x})] \tag{3.10}$$

The action S_Ω reproducing the right physics is given by

$$S_\Omega(\psi^*,\psi;A,\vec{W}) = \int_{M_3}\sqrt{|\eta|}d^3x\left[i\hbar\psi^* D_0\psi + \frac{\hbar^2}{2mc}\eta^{kl}(D_k\psi)^*(D_l\psi)\right] - \int_{\mathbb{R}} dx^0 H_I(x^0) \tag{3.11}$$

Quantization of the system with Feynman path integrals leads to the following formula for the partition function

$$\mathcal{Z}_\Omega(A,\vec{W}) = \int \mathcal{D}\psi^*\mathcal{D}\psi e^{iS_\Omega(\psi^*,\psi;A,\vec{W})/\hbar} \tag{3.12}$$

The functional $\ln\mathcal{Z}_\Omega(A,\vec{(}W)$ is the generating function of the connected time-ordered Green functions of the electromagnetic- and spin currents. At non-coinciding arguments,

$$\left\langle T\left[\prod_{i=1}^n j^{\mu_i}(x_i)\prod_{l=1}^m s_{A_l}^{\nu_l}(y_l)\right]\right\rangle_{a,w}^c = i^{n+m}\prod_{i=1}^n\frac{\delta}{\delta a_{\mu_i}(x_i)}\prod_{l=1}^m\frac{\delta}{\delta w_{\nu_l A_l}(y_l)}\ln\mathcal{Z}_\Omega(a,w) \tag{3.13}$$

where $<.>_{a,w}^c$ denotes the connected groundstate expectation functional in an external gauge field configuration (a,w). The effective action becomes

$$S_\Omega^{eff}(a,w) = \frac{\hbar}{i}\ln\mathcal{Z}_\Omega(a,w) \tag{3.14}$$

3.2 Scaling Limit

In the last part of the preceding subsection we defined the effective action on a connected finite domain, in this subsection the large scale behaviour of the system given by the effective action is studied. This is done by taking the system to the scaling limit, due to the assumption of strong cluster properties for the electromagnetic currents j this limit can actually be calculated. Furthermore we have restricted the analysis to the $U(1)$ part of the theory by setting $\mu_e = 0$. The scaling transformation maps elements $\xi\in\Omega$ into $\lambda\Omega$ that is

$$x = (x^0,\vec{x}) = \lambda(\xi^0,\vec{\xi}) \equiv \lambda\xi,\ 1\le\lambda<\infty \tag{3.15}$$

to make this transformation work the vectorpotentials have to be adjusted. The vector potential is split in two parts, an external potential a_c and a fluctuation potential $\tilde{a}$. This is done in a manner that the external vectorpotential on the transformed region $\lambda\Omega$ has a fixed strength, whereas the fluctuation potentials are fixed on Ω and therefore the strength of the fluctuation potentials decreases in the scaling-limit. That is

$$\begin{aligned} a_\mu^{(\lambda)}(x) &= a_{c,\mu}(x) + \tilde{a}_\mu^{(\lambda)}(x) \\ &= \lambda^{-1}[a_{c,\mu}(\xi;\lambda) + \tilde{a}_\mu(\xi)] \end{aligned} \tag{3.16}$$

The derivatives behave like λ^{-1}

$$\frac{\partial}{\partial x^\mu} = \lambda^{-1}\frac{\partial}{\partial \xi^\mu} \tag{3.17}$$

the scaling properties of a_μ where in fact chosen to give the covariant derivative the right scaling behaviour. That is scaling behaviour similar to the scaling behaviour of the derivatives.
The scaling-limit of the effective action S^* is defined by identifying $S^*_\Omega(a,w)$ with the relevant and marginal terms in the Laurent series in λ of the scaled effective action, that is the terms proportional to λ^{-D} for $D<0$ respectively $D=0$. The terms with $D>0$ are called irrelevant. The analysis starts with making a Taylor expansion of the effective action around a fixed background potential a_c, which yields

$$\begin{aligned}
S^{eff}_{\lambda\Omega}(a) &\equiv S^{eff}_{\lambda\Omega}(a,w=0) = \\
S^{eff}_{\lambda\Omega}(a_c) &- i\hbar\sum_{n=1}^{3}\frac{(-i)^n}{n!}\int_{(\lambda M_3)^n}\langle T[j^{\mu_1}(x_1)\dots j^{\mu_n}(x_n)]\rangle^c_{a_c} \\
&\times \tilde{a}_{\mu_1}(x_1)\dots\tilde{a}_{\mu_n}(x_n)dv(x_1)\dots dv(x_n) \\
&- \frac{i\hbar}{4!}\int_{(\lambda M_3)^4}\langle T[j^{\mu_1}(x_1)\dots j^{\mu_4}(x_4)]\rangle^c_{a_c+\alpha} \\
&\times \tilde{a}_{\mu_1}(x_1)\dots\tilde{a}_{\mu_4}(x_4)dv(x_1)\dots dv(x_4)
\end{aligned} \tag{3.18}$$

where $dv(x) = \sqrt{|\eta(x)|}d^3x$. The last term in this expansion, the remainder term, will turn out to be irrelevant and gives no contribution to the scaling limit. In the following the assumptions leading to the scaling limit are given, the term by term discussion of the expansion will be omitted and the result will be presented. For details see the paper by J. Fröhlich und M. Studer (loc.cit.).

Principle 3.1 (Incompressibility) *In the form of strong cluster properties of the connected green functions for currents incompressibility is put into the formalism*

$$\langle T[j^{\mu_1}(\lambda\xi_{i_1})\dots j^{\mu_n}(\lambda\xi_{i_n})]\rangle^c_{a_c} \to \varphi^{\mu_1\dots\mu_n}(\xi_1,\dots,\xi_n) \tag{3.19}$$

as $\lambda\to\infty$, where $\varphi^{\mu_1\dots\mu_n}$ is a local distribution, i.e. the support takes the following form

$$supp(\varphi^{\mu_1\dots\mu_n}) = \{\xi_1,\dots,\xi_n;\xi_1=\dots=\xi_n\} \tag{3.20}$$

The strong clustering properties are essential in calculating the scaling limit because of the scaling behaviour of the local distribution. A local distribution in three dimensions $\varphi^{\mu_1\dots\mu_n}(\xi_1,\dots,\xi_n)$ can be written as a sum of derivatives of produkts of $(n-1)$ 3-dimensional δ-functions. A 3-dimensional δ-function scales as λ^{-3} and thus $\varphi^{\mu_1\dots\mu_n}(\xi_1,\dots,\xi_n)$ scales as $\lambda^{-3(n-1)-a}$ where a is the number of derivatives.

Principle 3.2 *Only relevant and marginal terms contribute to the action S^*_Ω. That is if*

$$(\lambda^{2n}) \int \langle T[j^{\mu_1}(\lambda\xi_1)\dots j^{\mu_n}(\lambda\xi_n)]\rangle^c_{a_c}) \times \tilde{a}_{\mu_1}(\xi_1)\dots\tilde{a}_{\mu_4}(\xi_n)dv(\xi_1)\dots dv(\xi_n) \tag{3.21}$$

is of order λ^{-D} for $\lambda \to \infty$ with $D > 0$,then this term will not be displayed in the scaling limit.

Because of Principle [3.1] we are able to say $D \geq n-3$ which explains why the expansion is made only to fourth order.

Principle 3.3 ($U(1)$-gauge invariance and current conservation) *The effective action $S^{eff}_{\lambda\Omega}$ must be a gauge invariant functional of $\tilde{a} = a - a_c$*

$$S^{eff}_{\lambda\Omega}(\tilde{a} + d\tilde{\chi}; a_c + d\chi) = S^{eff}_{\lambda\Omega}(\tilde{a}; a_c) \tag{3.22}$$

from which follows

$$\nabla_{\mu_l}\varphi^{\mu_1\dots\mu_l\dots\mu_n}(\xi_1,\dots,\xi_l,\dots,\xi_n) = 0 \tag{3.23}$$

for $l = 1,\dots,n$ and here $\nabla_\mu = \frac{1}{\sqrt{|\eta|}}\partial(\sqrt{|\eta|}\cdot)$.

Using the above mentioned principles and rewriting the contributions in the language of differential forms one comes to the result

$$\begin{aligned} S^{eff}_{\lambda\Omega}(\tilde{a}) &\equiv S^{eff}_{\lambda\Omega}(\tilde{a}; a_c) \\ &= -\hbar\int_{M_3}(*j_c)\wedge\tilde{a} - \alpha\hbar\int_{M_3}\tilde{a}\wedge d\tilde{a} + \text{b.t.} \\ &= -\hbar\int_{M_3}(*j_c)\wedge a - \alpha\hbar\int_{M_3} a\wedge da + 2\alpha\hbar\int_{M_3} a\wedge da_c + \text{const} + \text{b.t.} \end{aligned} \tag{3.24}$$

where $\alpha = 4\pi\sigma$ and σ denotes the Hall-conductivity. J. Fröhlich and T. Kerler [FK91] showed that quantization of σ follows if one demands that there are excitations with Fermi-statistics and charge $\pm e$, that is electrons and holes. Using this assumption the Hall-conductivity, σ, takes the following values

$$\sigma = \pm\frac{1}{2l+1}\frac{e^2}{h} \tag{3.25}$$

here the non-simple filling fractions are not described, however for non-simple filling fractions the excitations are not Fermionic and have a charge which is not equal to $\pm e$. The other filling fractions can also be reconstructed in a similar way see for example the paper by J. Fröhlich and F. Zee [FZ91].

Chapter 4

Unique Vacuumstate, Clustering and Incompressibility

4.1 P_1CT symmetry in 2+1 dimensions

To construct a connection between uniqueness of the groundstate and strong clusterproperties we need a version of the PCT-theorem. The PCT-theorem in 3 +1 dimensions as originally proved by R. Jost [Jos65] is based on weak local commutativity and the properties of the holomorphy envelope for functions analytic on the forward tube domain. The forward tube domain is defined by

$$\begin{aligned} \mathcal{T}_n &= \mathcal{T}_n(\zeta_1, \ldots, \zeta_n) \\ &= \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^{Sn} : \zeta_i = \xi_i - i\eta_i\} \end{aligned} \tag{4.1}$$

where S is the dimension of space-time, $i = 1, \ldots, n$ and $\eta_i \in V_+$ the forward cone

$$V_+ = \{\xi : (\xi, \xi) > 0, \xi^0 > 0\} \tag{4.2}$$

ξ is a vector in S-dimensional spacetime. Weak local commutativity (WLC), in real neighbourhood of real point (Jost point) in terms of Wightman distributions for scalar fields is

$$\begin{aligned} \langle 0 \,|\phi(x_1) \ldots \phi(x_n)|\rangle &= \langle 0 \,|\phi(x_n) \ldots \phi(x_1)|\rangle \\ &\Leftrightarrow \\ W(\xi_1, \ldots, \xi_{n-1}) &= W(-\xi_{n-1}, \ldots, -\xi_1) \end{aligned} \tag{4.3}$$

If weak local commutativity holds it can be generalized to the extended tube

$$\mathcal{T}_n' = \bigcup_{\Lambda \in L_+(C)} \Lambda \mathcal{T}_n \tag{4.4}$$

where $L_+(C)$ is the complex Lorentz group with determinant +1. In 3 + 1 dimensions these conditions lead to PCT invariance that is invariance under the combined action of the following operators

- P space inversion $\xi^i \to -\xi^i$, $i = 1, 2, 3$
- C charge conjugation
- T time inversion $\xi^0 \to -\xi^0$

In 2 + 1 dimensions PCT invariance does not apply, however A.C. Manoharan [Man68] showed that another invariance does hold. In 3 + 1 dimensions PCT invariance is equivalent to P_1CT in 2 + 1 dimensions these invariances are not equivalent and P_1CT invariance does apply where PCT now fails. The P_1CT operator is the combined action of C, T and the P_1 operator, where

- P_1 space inversion in one coordinate $\xi^1 \to -\xi^1$, $\xi^2 \to \xi^2$

In 2 + 1 dimensions P_1CT invariance is the consequence of the same assumptions that led to PCT invariance in 3 +1 dimensions. P_1CT has been proved for scalar Wightman fields only, a more general version was investigated by B. Kuckert [Kuc94] he obtained within the algebraic framework a P_1CT symmetry for fields localized in spacelike cones using a weaker version of P_1CT symmetry defined on double cones. Kuckert also found that P_1CT symmetry allows fractional statistics, which is important if one applies P_1CT to a field theory describing the Fractional Quantum Hall-effect.

4.2 Clusterproperties and uniqueness of the vacuumstate

In 1962 H.J. Borchers proved that irreducibility of a Wightman field describing bosons implies uniqueness of the vacuumstate and is equivalent to clustering. We will follow his approach but using P_1CT-symmetry instead of PCT-symmetry to construct his theorem in 2 + 1 dimensions. We work with Wightman functionals for a full treatment of these functionals we refer to [SW64] and [Bor62]. If Ω is any vacuum-state we can define a P_1CT-operator θ by

$$\begin{aligned} \theta\Omega &= \Omega \\ \theta A(x)\Omega &= A(x')\Omega \\ \theta A(x_1)A(x_2)\Omega &= A(x_1')A(x_2')\Omega \\ x'^0 &= -x^0, x'^1 = -x^1, x'^2 = x^2 \end{aligned} \tag{4.5}$$

θ maps the subspace of invariant vectors onto itself. If we assume that there exist other vacuum states one can find a base in the subspace Ω_i so that $\theta\Omega_i = \Omega_i$. This is a consequence of $\theta^2 = 1$ [AG54]. If we now construct a new vacuum state from the Ω_i

$$\Omega_{\alpha\beta} = \alpha\Omega_1 + \beta\Omega_2 \tag{4.6}$$

with

$$|\alpha|^2 + |\beta|^2 = 1, \frac{\alpha}{\alpha^*} \neq \frac{\beta}{\beta^*} \tag{4.7}$$

and define $\theta_{\alpha\beta}$ as we defined θ but with the state Ω replaced by $\Omega_{\alpha\beta}$. The P_1CT-theorem is also applicable to the state $\Omega_{\alpha\beta}$ and $\Omega_{\alpha\beta}$ is a cyclic state. And we have

$$\begin{aligned} \theta\theta_{\alpha\beta}\Omega_{\alpha\beta} &= \theta(\alpha\Omega_1 + \beta\Omega_2) \\ &= \alpha^*\Omega_1 + \beta^*\Omega_2 \neq \rho\Omega_{\alpha\beta} \end{aligned} \tag{4.8}$$

where ρ is scalar. As a direct consequence of the definition of θ we have $\theta\theta_{\alpha\beta}A(x)\theta_{\alpha\beta}\theta = A(x)$ and we see that the operator $\theta\theta_{\alpha\beta}$ commutes with the field algebra. If the field algebra is irreducible then

$$\theta\theta_{\alpha\beta} = \rho \tag{4.9}$$

equation [4.9] and equation [4.8] are in contradiction leading to the following theorem

Theorem 4.1 *If W is a Wightman functional describing an irreducible field the subspace of zero-energy-states is one-dimensional*

We are now ready to present the following lemma

Lemma 4.1 *If $U(a, \Lambda)$ is a continuous unitary representation of the inhomogeneous Lorentz-group and Ψ is any state in $\mathcal{H}$ then we have*

$$w - \lim_{\rho\to\infty} U(a, \Lambda)\Psi = P_0\Psi \tag{4.10}$$

where the limit is taken in the weak topology and P_0 is the projection operator onto the translation invariant states.

The proof of this lemma can be found in [Bor62] and applies without change in $2 + 1$ dimensions, however Borchers stated that in $1 + 1$ dimensions it does not hold. With this lemma and the theorem we can prove following theorem

Theorem 4.2 *In order that a Wightman functional describes an irreducible field it is necessary and sufficient that W fulfills the equation*

$$\lim_{\rho\to\infty} W(g \times (\rho a, 1)h) = W(g)W(h) \tag{4.11}$$

where $a \neq 0$

Proof: If the field is irreducible one can easily see from the preceding theorem [4.1] and lemma [4.1] using $W(g) = \langle\Psi_0, \Psi(g)\rangle$ that the equation is fulfilled. If the condition of the theorem is fulfilled we have

$$w - \lim_{\rho\to\inf} U(\rho a, 1)\Psi(h) \to \Psi_0(\Psi_0, \Psi(h)) \tag{4.12}$$

where Ψ_0 is the cyclic state. From Lemma [4.1] we find that the subspace of invariant states is one-dimensional. If the field is reducible the space of invariant states has more than one dimension. Which proves the theorem. Using theorem [4.1]and [4.2] we can now show how the implication in theorem [4.1] can be made into an equivalence, thereto we have to proof that uniqueness of the vacuum state implies irreducibility of the Wightman functional. We can use an other theorem proved by Borchers [Bor62] which we checked to be valid if the P_1CT symmetry holds

Theorem 4.3 *Every Wightman-functional can be decomposed into a sum or integral of inequivalent Wightman-functionals describing only irreducible fields*

A consequence of this theorem is that a vacuum state can be decomposed as follows

$$\Psi_0 = \sum_i \rho_i \Psi_{i0}, \sum |\rho_i|^2 = 1 \tag{4.13}$$

this implies that if Wightman functional is reducibel the space of vacuumstates is more than one dimensional. Resuming we see that the following three properties are equivalent

Theorem 4.4

- *Irreducibility of the field described by a Wightman functional W*
- *Uniqueness of the vacuumstate*
- *Clusterproperty :* $\lim_{\rho\to\infty} W(g \times (\rho a, 1)h) = W(g)W(h)$

4.3 Different concepts of incompressibility

The insights of Borchers enable us to relate the concept of incompressibility proposed by Laughlin to the clusterproperty used by Fröhlich und Studer to implement incompressibility. Reconstructing Borchers proofs in (2+1)-dimensions we learned that for scalar Wightman-fields the uniqueness of the groundstate is equivalent to cluster-properties for the fields. It is possible to show that clustering used by Borchers gives rise to local distributions as used by Fröhlich und Studer, however one has to keep in mind the differences between the systems the clusterproperties are applied to. During scaling all observables and the system itself are changing where the cluster property used by Borchers applies to one observable translated away from another observable. But if we are able to proof the clusterproperties Borchers uses, then incompressibility as used Fröhlich und Studer would seem not unlikely.
However the Wightman functionals used in proving this statement deal with fields that have a "commutator structure" as opposed to an "anti-commutator structure", where Laughlin's theory and the theory of Fröhlich und Studer deals with Fermions and hence with anti-commutator structures. The idea to circumvent this problerm is to look at the representations induced by the Wightman functional on a restricted set of field operator products: the even field operator

products. Among those are the currents used by Fröhlich und Studer in their definition of incompressibility.
This is a serious option because Powers and Doplicher [PD57] showed that a representation of a CAR-algebra can be restricted to a subalgebra of even field operator products thus giving a faithful representation of a commuting structure to which Borchers theorem can be applied. That uniqueness of the groundstate found in Laughlin's theory holds in the thermo-dynamic limit is not clear. Araki and Wyss [AW64] showed that a system of free electrons with a fixed density becomes irreducible when taken to the thermodynamic limit.
Furthermore its difficult to say whether or not a version of Borchers-theorem for fields satisfying fractional statistics is needed. The statistical nature of the excitations in the system is fractional, however underneath we find electrons which have Fermi-statistics. If a version of Borchers theorem for fields satisfying fractional statistics is needed another formalism then Wightman field theory in its original form is probably needed because fields carrying representations of the braid-group are up to now defined on unbounded regions of space-time where Wightman fields are defined on bounded space-time regions.
Fröhlich and Studer assert that the existence of an energy gap implies clusterproperties, in the context of Wightman fields such a gap is not enough to ensure clustering and an additional property is needed which can be found by using the equivalence of the statements in theorem 4.4 obtained in the previous section. Incompressibility implies a gap because the system should be stable against small changes, incompressibility furthermore implies uniqueness of the lowest energy states otherwise a change is possible at no cost. However we can now prove clustering taking these two properties as a starting point for Bosonic Wightman fields in (2+1)-dimensions.

Chapter 5

Algebraic Qauntum Field Theory and Duality

5.1 Basic concepts of Algebraic QFT

Algebraic quantum field theory explores the relationship between fields, observables and the gauge group within an algebraic frame work. The ultimate goal is to construct a field algebra and the gauge group from an algebra of observables. The starting points for an analysis in AQFT are some very general principles of quantum physics formulated in algebraic language, these principles are formulated as assumptions and are sometimes presented as axioms but one has to keep in mind that they are still subject of discussion. We will enter this discussion when analyzing different duality concepts and their consequences. We present here the assumptions made by Doplicher Haag and Roberts in their paper [DHR69b] after that we will briefly sketch the construction of the field algebra skipping some technicalities only to be able to present a theorem about the statistics of the field operators in (3+1)-spacetime dimensions. Then we present a modified version of assumption [5.4] proposed and motivated by J. Fröhlich, F. Gabbiani and P.A. Marchetti in [FGM] leading to anyon statistics in (2+1)-spacetime dimensions and we establish a connection to the preceding chapters of this report.

Assumption 5.1 *There is a correspondence between closed double cones in Minkowski space and von Neumann algebras on a Hilbert space* $\mathcal{H}_0$

$$\mathcal{O} \to \mathfrak{A}(\mathcal{O}), \mathcal{O} \in \mathcal{K} \tag{5.1}$$

If $\mathcal{O}_1$,$\mathcal{O}_2$ *and* $\mathcal{O}_1 \subset \mathcal{O}_2$, *then* $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. *If* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are spacelike* $\mathfrak{A}(\mathcal{O}_1)$ *and* $\mathfrak{A}(\mathcal{O}_2)$ *commute. The* C^**-algebra of quasilocal observables* $\mathfrak{A}$ *is defined as the uniform closure of the union of all local observable algebra's*

$$\mathfrak{A} = \overline{\bigcup_{\mathcal{O}\in\mathcal{K}} \mathfrak{A}(\mathcal{O})}^{n} \tag{5.2}$$

and is assumed to be weakly dense in the set of all bounded operators on $\mathcal{H}_0$. $\mathfrak{A}(\mathcal{O}')$ *denotes the* C^**-subalgebra of* $\mathfrak{A}$ *generated by all* $\mathfrak{A}(\mathcal{O}_i)$ *with* $\mathcal{O}_i \in \mathcal{K}$ *spacelike to* $\mathcal{O}$. *For general unbounded regions* $\mathcal{S}$ *we can define in similar way*

$$\mathfrak{A} = \overline{\bigcup_{\substack{\mathcal{O}\in\mathcal{S}\\ \mathcal{O}\in\mathcal{K}}} \mathfrak{A}(\mathcal{O})}^{n} \tag{5.3}$$

where a closed double cone is the intersection of forward lightcone with a backward light cone with non-void interior and a von Neumann-algebra $\mathcal{A}$ is an algebra which is weakly closed, that $\overline{\mathcal{A}}^{w} = \mathcal{A}$.

Assumption 5.2 *The Poincaré group in (3+1)-spacetime dimensions* $\mathcal{L}$ *is represented by automorphisms* α_L *of* $\mathfrak{A}$ *,*$L \in \mathcal{L}$. α_L *transforms a local subalgebra* $\mathfrak{A}(\mathcal{O})$ *onto the subalgebra of the transformed region* $\mathfrak{A}(L\mathcal{O})$.

Assumption 5.3 *There is a strongly continuous unitary representation* U_0 *of* $\mathcal{L}$ *on* $\mathcal{H}_0$ *implementing the automorphisms* α_L, $L \in \mathcal{L}$*:*

$$U_0(L)AU_0(L)^{-1} = \alpha_L(A),\ L \in \mathcal{L} \tag{5.4}$$

The representation U_0 *satisfies the following spectrum condition: the energy momentum operator has its spectrum confined to* $E \geq 0$, *the eigenvalue zero being nondegenerate and corresponding to the vacuum state* ω_0 *which is represented by a vector* $\Omega \in \mathcal{H}_0$.

$$\omega_0(A) = (\Omega, A\Omega),\ A \in \mathfrak{A} \tag{5.5}$$

we have seen that this assumption is satisfied if we replace the vacuum state by the groundstate of the FQHE

Assumption 5.4 *Duality holds for each double cone:*

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')',\ \mathcal{O} \in \mathcal{K} \tag{5.6}$$

where the $\mathfrak{A}(\mathcal{O}')'$ *denotes the subalgebra of* $\mathfrak{A}$ *commuting with* $\mathfrak{A}(\mathcal{O}')$.

In order to present the theorem on statistics based on the axioms we presented we need to define some concepts.

Definition 5.1 (Localized Automorphisms) *An automorphism* γ *of* $\mathfrak{A}$ *is called localized in a region* $\mathcal{O}$ *if*

$$\gamma(A) = A\ for\ A \in \mathfrak{A}(\mathcal{O}') \tag{5.7}$$

the group of automorphisms localized in $\mathcal{O}$ *is denoted* $\Gamma(\mathcal{O})$ *and the algebra of all localized automorpisms is*

$$\Gamma = \bigcup_{\mathcal{O}\in\mathcal{K}} \Gamma(\mathcal{O}) \tag{5.8}$$

A subset Γ_c of Γ is asociated with covariant representations. An automorpism γ is in Γ_c if the representation asociated with γ is covariant. The representation asociated with γ is

$$\pi_\gamma = \pi_0 \circ \gamma \tag{5.9}$$

and π_γ is called covariant if there is strongly continuous representation U_π on the Hilbert space $\mathcal{H}_\pi$ asociated with π_γ implementing all automorphisms α_L, $L \in \mathcal{L}_+^\dagger$

$$\pi(\alpha_L(A)) = U_\pi(L)\pi(A)U_\pi(L)^{-1} \tag{5.10}$$

Definition 5.2 *There is a subalgebra of $\mathfrak{A}(\mathcal{O})$ constituted by unitary operators $\boldsymbol{\mathcal{A}}(\mathcal{O})$ and there is a map σ from*

$$\boldsymbol{\mathcal{A}} = \bigcup_{\mathcal{O}\subset\mathcal{K}} \boldsymbol{\mathcal{A}}(\mathcal{O})$$

to the localized automorphisms

$$\sigma_U(A) = UAU^{-1},\ for U \in \boldsymbol{\mathcal{A}}, A \in \mathfrak{A} \tag{5.11}$$

the image of $\boldsymbol{\mathcal{A}}$ under σ is the normal subgroup $\mathcal{J}$ of Γ_C. Thus we can define the quotient group $\Gamma_C/\mathcal{J}$.

A sector is a set of unitary equivalent representations, two representations $\pi_0\circ\gamma_1$ $\pi_0 \circ \gamma_2$ are called unitary equivalent if

$$\pi_0(\gamma_1(A)) = V\pi_0(\gamma_2(A))V^{-1},\ A \in \mathfrak{A} \tag{5.12}$$

where V is a unitary operator on $\mathcal{H}_0$.

Lemma 5.1 *$\pi_0 \circ \gamma_1$ and $\pi_0 \circ \gamma_2$ are unitary equivalent if and only if $\gamma_1\gamma_2^{-1} \in \mathcal{J}$*

In [DHR69a] it was proved that the sectors are in one to one correspondence with the dual of the gauge group, that is the group $\hat{G}$ of characters of G, therefore we can make the following identification

$$\hat{G} = \Gamma_C/\mathcal{J} \tag{5.13}$$

It is this identification that motivates the following definition of the field algebra

$$\boldsymbol{\mathfrak{F}}_0 = \boldsymbol{\mathcal{A}} \times_s \Gamma_C/\mathcal{J} \tag{5.14}$$

where $\times_s$ denotes a semidirect product and the product on the algebra $\boldsymbol{\mathfrak{F}}_0$ is defined by

$$\Psi\Psi' = (Us(\xi)(U'), \xi\xi') \tag{5.15}$$

where s is a map from a sector $\xi \in \Gamma_C/\mathcal{J}$ to Γ_C. Under the Klein-transformation [B.1] which is one to one $\boldsymbol{\mathfrak{F}}_0$ transforms to a $\boldsymbol{\mathfrak{F}}$ satisfying the properties required of a field algebra

- $\boldsymbol{\mathcal{A}}$ is a normal subgroup of $\boldsymbol{\mathfrak{F}}$

- the homomorpism σ of $\boldsymbol{\mathcal{A}}$ onto$\mathcal{J}$ extends to a homomorphism $\Psi \to \sigma_\Psi$ of $\mathfrak{F}$ onto Γ_C, satisfying
  1. $\sigma_\Psi(U) = \Psi U \Psi^{-1}$ if $\Psi \in \mathfrak{F}$,$U \in \boldsymbol{\mathcal{A}}$
  2. Kernel $\sigma = Kernel(\sigma|\boldsymbol{\mathcal{A}})$

This $\mathfrak{F}$ satifies the following theorem

Theorem 5.1 (Normal Statistics) *Let $\Psi_1 \in \mathfrak{F}(\mathcal{O}_1)$,$\Psi_2 \in \mathfrak{F}(\mathcal{O}_2)$, with $\mathcal{O}_1$ spacelike to $\mathcal{O}_2$; then*

$$\Psi_1\Psi_2 = \pm\Psi_2\Psi_2 \tag{5.16}$$

where the minus sign holds only if both fields are Fermi fields.

This theorem shows that in Algebraic Quantum Field Theory in (3+1)-spacetime dimensions and assuming Haag duality only Bosonic and Fermionic fields are allowed and other forms of statistics are excluded.

5.2 Weak duality

A modification of duality in (2+1)-dimensional spacetime is necessary because localization of the observables and fields in double cones together with Haag duality does not permit fractional statistics which is observed in the (2 + 1)-dimensional spacetime of Quantum Hall systems. From Laughlin we learned that making an excitation with a charge e^* is accompanied by transporting charge $-e^*$ to infinity thus making the excitation non-local therefore we expect to need a modification of duality reflecting the non-local character of the exciations. The modification discussed here is based on the behaviour of Greens functions in Euclidian Abelian $U(1)$ Higgs-models in 2+1 dimensions with an Chern-Simons term added and satisfying the Osterwalder Schrader axiom. The modifications were proposed by J. Fröhlich F. Gabbiani and P.A. Marchetti in [FGM]. The underlying theory of this Higgs-model can be found in [FM89] and [Has79].
The key observation extracted from this theory is that the Green functions connecting field algebra's asociated with different spacelike cones can be aproximated by Wilson operators which can be localized in simple regions. We need to explain this statement:

Definition 5.3 (Fields Asociated with Space-like Cones) *A field algebra $\mathfrak{F}(\mathcal{C})$ is asociated with a spacelike cone $\mathcal{C}$ which is a region in spacetime of the following form*

$$\mathcal{C} = a + \bigcup_{\lambda>0} \lambda\mathcal{O} \tag{5.17}$$

where $\mathcal{O}$ is an open double cone whose closure is spacelike to the origin of spacetime $0 \in \mathbb{M}^3$. The cone is called a space-like cone with apex a. A region $\mathcal{S} \subseteq \mathbb{M}^3$ is called simple iff $\mathcal{S}$ is a space-like cone or $\mathcal{S}$ is the space-like complement of a space-like cone.

Definition 5.4 (Wilson Operator) *A Wilson operator is bounded operator of the following form*

$$W(j) = e^{i\int d^3\xi j^\mu(\xi)A_\mu(\xi)} \tag{5.18}$$

where j is a vector valued function with support in a ring surrounding a loop $\mathcal{L}$ located in a spacelike surface. j satisfies the equation $\partial_\mu j^\mu = 0$. The Wilson operator is gauge invariant and $W(j) \in \mathfrak{A}(\mathcal{S})$ if $\mathcal{L} \subset \mathcal{S}$ where $\mathcal{S}$ is simple.

in the above statement we ignored the possible presence of Higgs-field to simplify our treatment. This is probably a correct approach because Hasenfratz noted that the presence of Higgsfields in not essential for the form of the Green-functions in the abelian Higgsmodel and the Higgs-fields are not present in Chern-Simons theory describing Quantum Hall-effects.
Consider a state in a sector n with an anyon localized in a space-like cone $\mathcal{C}_1$ containing the origin. We can define a space-like cone $\mathcal{C}_2$ by rotating $\mathcal{C}_1$ through an angle θ. Let $C_n(\theta)$ denote the operation of transporting the anyon from $\mathcal{C}_1$ to $\mathcal{C}_2$ along a sequence of cones $\mathcal{C}(\theta')$ with $0 \leq \theta' \leq \theta$, where $\mathcal{C}(\theta')$ is obtained from $\mathcal{C}_1$ by a rotation through an angle θ'

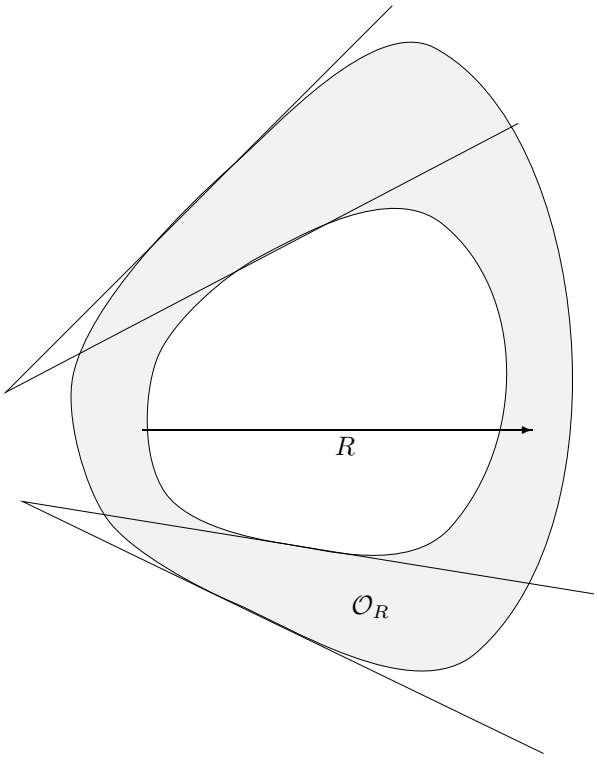


Figure 5.1: Regions $\mathcal{O}_R$

, then we can asociate $C_n(\theta)$ with an approximation in the weak operator topology by Wilson operators $W(j_R)$ localized in regions $\mathcal{O}_R(\theta)$. See also figure [5.2]. Where R tends to infinity and the currents j_R are appropriatedly chosen. If we now take a simple region

$$\mathcal{S} \supseteq \bigcup_{0 \leq \theta' \leq \theta} \mathcal{C}(\theta') \tag{5.19}$$

we have

$$C_n(\theta) \in \overline{\mathfrak{A}(\mathcal{S})_n}^w \tag{5.20}$$

where $^{-w}$ denotes the closure in the weak operator topology. If we now define $\mathcal{S}_{12}$ to denote the causal complement of the space-like cones $\mathcal{C}_1$ and $\mathcal{C}_2$ it follows from Einstein causality for observables that

$$C_n(\theta) \in \mathfrak{A}(\mathcal{S}_{12})'_n \tag{5.21}$$

and thus

$$C_n(\theta) \in \mathfrak{A}(\mathcal{S}_{12})'_n \cap \overline{\mathfrak{A}(\mathcal{S})_n}^w \tag{5.22}$$

.
We can also rotate the cones in the opposite direction, then we need a sequence of cones $\mathcal{C}(\theta')$, where $\mathcal{C}(\theta')$ denotes $\mathcal{C}_1$ rotated through an angle θ', with $2\pi - \theta \leq \theta' \leq 0$. And denote the operation of transporting the anyon along this sequence of cones by $C_n(\theta - 2\pi)$. Now the relations [5.21] and [5.22] hold for $C_n(\theta - 2\pi)$ provided we replace $\mathcal{S}$ by $\mathcal{S}'$

$$\mathcal{S}' = \supseteq \bigcup_{\theta - 2\pi \leq \theta' \leq 0} \mathcal{C}(\theta') \tag{5.23}$$

what is important in view of the following is that the operators $C_n(\theta)$ and $C_n(\theta - 2\pi)$ are not necessarily equivalent as one might expect in case of normal statistics.
We are now ready to formulate the modified duality proposition proposed by J.Fröhlich, F.Gabbiani and P.A. Marchetti. This form is motivated by identifying the Greens-functions $C_n(\theta)$ in the Abelian Higgs-model with the transport operators in the algebraic frame, we need to reformulate some notions to make this identification

Definition 5.5 (Localizable Representations) *A representation $(\pi, \mathcal{H})$ of $\mathfrak{A}$ is said to be localizable in cones relative to a vacuum representation $(\pi_0, \mathcal{H})$ if for any space-like cone $\mathcal{C}$ there exists an isometry $V_\mathcal{C}$ from $\mathcal{H}$ onto $\mathcal{H}_0$ such that*

$$V_\mathcal{C} \pi(A) = \pi_0(A) V_\mathcal{C} \text{ for } A \in \mathfrak{A}^c(\mathcal{C}) \equiv \mathfrak{A}(\mathcal{C})' \cap \mathfrak{A} \tag{5.24}$$

The family of all such representations of $\mathfrak{A}$ is denoted by $\mathcal{L}_{\pi_0}$.

Using this isometries $V_\mathcal{C}$ we can define a representation $\gamma_\mathcal{C}$ of $\mathcal{H}_0$ equivalent to π by the following automorphism localized in $\mathcal{C}$

$$\gamma_\mathcal{C}(A) = V_\mathcal{C} \pi(A) V_\mathcal{C}^{-1}, \text{ for all } A \in \mathfrak{A} \tag{5.25}$$

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are arbitrary space-like cones and with a sector π are asociated two unitary equivalent representations $\gamma_{\mathcal{C}_1}$ and $\gamma_{\mathcal{C}_2}$ of $\mathfrak{A}$. Then there exists a unitary transport operator $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$ on $\mathcal{H}_0$ such that

$$\Gamma(\mathcal{C}_1, \mathcal{C}_2)\gamma_{\mathcal{C}_2} = \gamma_{\mathcal{C}_2}\Gamma(\mathcal{C}_1, \mathcal{C}_2) \text{ for all } A \in \mathfrak{A} \tag{5.26}$$

Proposition 5.1 (Weak Duality) • *Let γ be a representation of $\mathfrak{A}$ on $\mathcal{H}_0$ localized in a space-like cone $\mathcal{C}$. Then for any space-like simple domain $\mathcal{S} \supseteq \mathcal{C}$*

$$\gamma(\mathfrak{A}(\mathcal{S})) \subseteq \overline{\mathfrak{A}(\mathcal{S})}^w \tag{5.27}$$

- *If $\gamma_{\mathcal{C}_1}$ and $\gamma_{\mathcal{C}_2}$ are two unitary equivalent representations of $\mathfrak{A}$ on $\mathcal{H}_0$ localized in space-like cones $\mathcal{C}_1$ respectively $\mathcal{C}_2$ and $\mathcal{S}$ is a simple region containig $\mathcal{C}_1 \cup \mathcal{C}_2$ then there is a unitary intertwiner $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$ such that*

$$\Gamma(\mathcal{C}_1, \mathcal{C}_2)\gamma_{\mathcal{C}_2} = \gamma_{\mathcal{C}_2}\Gamma(\mathcal{C}_1, \mathcal{C}_2) \text{ for all } A \in \mathfrak{A} \tag{5.28}$$

and

$$\Gamma(\mathcal{C}_1, \mathcal{C}_2) \in \mathcal{B}(\mathcal{C}_1 \cup \mathcal{C}_2) \cup \overline{\mathfrak{A}(\mathcal{S})}^w \tag{5.29}$$

where

$$\begin{aligned} \mathcal{B}(\mathcal{C}_1 \cap \mathcal{C}_2) &= (\mathfrak{A}(\mathcal{C}'_1) \cup \mathfrak{A}(\mathcal{C}'_2))' \\ \mathcal{B} &= \mathfrak{A}(\mathcal{S}')' \end{aligned} \tag{5.30}$$

this inclusion is motivated by the identification of $C_n(\theta)$ with $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$

- *If $\mathcal{C}$ and $\mathcal{S}$ are simple regions and $\mathcal{S} \supseteq \mathcal{C}$ and $\mathcal{S}'$ contains some space-like cone then*

$$\mathcal{B}(\mathcal{C}) \cup \overline{\mathfrak{A}(\mathcal{S})}^w = \overline{\mathfrak{A}(\mathcal{C})}^w \tag{5.31}$$

this condition is in its implications equivalent to requiring the Hilbert space to be a direct sum of irreducible representations

For the occurence of fractional statistics it is essential that the space Σ_3 of space-like cones in $\mathbb{M}^3$ and the rotation subgroup of the Poincaré-group have non-trivial topologies which give rise to the fundamental group

$$\pi_1(S_1) = \mathbb{Z} \tag{5.32}$$

where S^1 denotes the circle. The space Σ_3 is difficult to handle, there is however a subgroup of Σ_3 which can be parametrized by three numbers and is equivalent to it with respect to the fundamental group. Let C be an open convex cone in the time $t = 0$ plane of $\mathbb{M}^3$, whose boundary, ∂C consist of two rays emanating from the apex x of C. Then C can be completely described by its apex x, the opening angle between the two rays ϵ and the asymptotic angle α, see figure [5.2], which is the angle between a ray bisecting C into two equal pieces and the x^1-axis. We let $\mathcal{C}$ denote the causal completion of C. That is $\mathcal{C} = (C')'$ which is also completely defined by the above mentioned parameters. Thus the cones in this

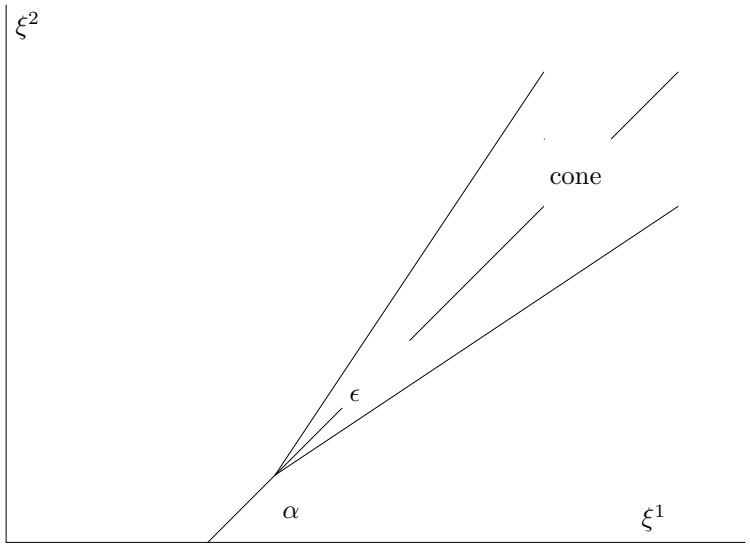


Figure 5.2: Asymptotic Direction of C

subgroup are completely specified by elements of the space $\mathcal{R}^2 \times (0, \pi) \times S^1$ and the fundamental group of this space is $\pi_1(S_1)$.
In (2+1)-dimensions the Poincaré group is

$$SO(2,1) \times_s \mathbb{R}^3 \tag{5.33}$$

where $SO(2,1)$ denotes the Lorentz group and $\mathbb{R}^3$ the group of translations, $\times_s$ gives the composition asociated with applying one Poincaré transformation after the other. That is for $L_i = (\Gamma_i, a_i) \in SO(2,1) \times_s \mathbb{R}^3,\ i = 1, 2$ we have the following composition law

$$L_1 \times_s L_2 = (\Gamma_1\Gamma_2, a_1 + \Gamma_1 a_2) \tag{5.34}$$

The Lorentz-group $SO(2,1)$ is homeomorphic to $\mathbb{R}^2 \times S^1$, that is the composition of a rotation in two dimensional space with a Lorentz-boost along a fixed axis. The fundamental group of the Lorentz group is therefore equivalent to the fundamental group of S^1
The non-trivial topology of the Σ_3 and the Poincaré group reflects the fact that the space of paths describing the interchange of fields falls apart in a set of

equivalence classes, where equivalence means that two paths can be continiuously transformed into each other. The inequivalence of paths in different classes leads to path dependence for the asociated interchange operators, which as a consequence carry a representation of the braid group.

5.3 Weakly Localized Fields

We will now introduce the machinery needed to evaluate statistics under the duality assumptions of the preceding section. We leave out the proofs as they can be found in [FGM] and [BF82] and restrict ourselves to those parts of the theory which are necessary to understand the implications of the duality proposition.
In order to define the product of two representations we need an auxilary algebra $\mathcal{B}^{\mathcal{C}_a}$ and an algorythm for extending a localized automorphism to this algebra. The algebra $\mathcal{B}^{\mathcal{C}_a}$ is defined by

$$\mathcal{B}^{\mathcal{C}_a} = \bigcup_{x\in\mathbb{M}^3} \overline{\mathfrak{A}((\mathcal{C}_a + x)')}^{w-n} \tag{5.35}$$

where $\mathcal{C}_a$ denotes a space-like cone and $\mathcal{C}_a + x$ denotes a translate of $\mathcal{C}_a$ along a vector $x \in \mathbb{M}^3$ If we now choose a representation $\gamma_{\mathcal{C}}$ localized in a space-like cone $\mathcal{C}$ then there exists an equivalent representation $\gamma_{\mathcal{C}_1}$ localized in $\mathcal{C}_1 \subseteq \mathcal{C}_a + x$ for some $x \in \mathbb{M}^3$. We know there is an intertwiner $\Gamma(\mathcal{C}, \mathcal{C}_1)$ intertwining $\gamma_{\mathcal{C}}$ and $\gamma_{\mathcal{C}_1}$

$$\gamma_{\mathcal{C}}(A) = \Gamma(\mathcal{C}, \mathcal{C}_1)\gamma_{\mathcal{C}_1}(A)\Gamma(\mathcal{C}, \mathcal{C}_1)^* \text{ for } A \in \mathfrak{A} \tag{5.36}$$

,and we have for $A \in \overline{\mathfrak{A}((\mathcal{C}_a + x)')}^{w}$, $\gamma_{\mathcal{C}_1}(A) = A$

$$\gamma_{\mathcal{C}} = \Gamma(\mathcal{C}, \mathcal{C}_1)A\Gamma(\mathcal{C}, \mathcal{C}_1) \tag{5.37}$$

and we can extend $\gamma_{\mathcal{C}}$ to $\overline{\mathfrak{A}((\mathcal{C}_a + x)')}^{w}$ by the following definition

$$\gamma_{\mathcal{C}}^{\mathcal{C}_a}(B) \equiv \Gamma(\mathcal{C}, \mathcal{C}_1)B\Gamma(\mathcal{C}, \mathcal{C}_1)^* \text{ for all } B \in \overline{\mathfrak{A}((\mathcal{C}_a + x)')}^{w} \tag{5.38}$$

The above construction can be done for all $x \in \mathbb{M}^3$ and defines an extension of $\gamma_{\mathcal{C}}$ to $\gamma_{\mathcal{C}}^{\mathcal{C}_a}$ defined on the auxilary algebra $\mathcal{B}^{\mathcal{C}_a}$. Using this extension and its properties (see proposition 5.1 in [FGM]) we can define a product of representations.

Definition 5.6 (Product of Representations) *The set of representations $\gamma_{\mathcal{C}} \in \mathcal{L}_{\pi_0}$ of $\mathfrak{A}$ on $\mathcal{H}_0$ and satisfying the first two conditions of the weak duality proposition and are localized in the space-like cone $\mathcal{C}$ is denoted by $\mathcal{L}_{\mathcal{C}}$. Choose an auxilary cone $\mathcal{C}_a \subseteq \mathcal{C}'$. For γ_1 and γ_2 in $\mathcal{L}_{\mathcal{C}}$ we define a product by*

$$\gamma_1 \circ \gamma_2(A) = \gamma_1^{\mathcal{C}_a}(\gamma_2(A)), \text{ for all } A \in \mathfrak{A} \tag{5.39}$$

This product has the following properties:

Theorem 5.2 *1. If $\gamma_1, \gamma_2 \in \mathcal{L}_{\mathcal{C}}$ then $\gamma_1 \circ \gamma_2 \in \mathcal{L}_{\mathcal{C}}$.*

2. $\gamma_1 \circ \gamma_2|_{\mathfrak{A}}$ does not depend on the choice of the auxilary cone $\mathcal{C}_a \subset \mathcal{C}'$.

3. If $\hat{\gamma}_i$ is equivalent to γ_i for $i = 1, 2$, and the $\hat{\gamma}_i$ belong to $\mathcal{L}_{\hat{\mathcal{C}}}$ for some space-like cone $\hat{\mathcal{C}}$ then $\gamma_1 \circ \gamma_2$ is equivalent to $\hat{\gamma_1} \circ \hat{\gamma_2}$

As a consequence of Gelfand-Naimark-Segal construction we can asociate with each representation π a Hilbert-space $\mathcal{H}_\pi$ containing a cyclic vector ξ_π. The total Hilbert-space of the theory can then be defined as

$$\mathcal{H}_{\rm tot} = \bigoplus_{[\gamma]:\gamma\in\mathcal{L}_{\pi_0}} \mathcal{H}_{[\gamma]} \tag{5.40}$$

where $[\gamma]$ denotes the unitary equivalence class represented by γ and $\mathcal{H}_{[\gamma]} \equiv \omega_\prime \circ \gamma$ where ω_0 denotes the vacuum-state $\omega_0(A) = \langle\Omega|A|\Omega\rangle$, where Ω is the vacuum-state vector in $\mathcal{H}_0$. The total Hilbert-space carries a representation $\pi_{\rm tot}$ of $\mathfrak{A}$

$$\pi_{\rm tot} = \bigoplus_{[\gamma]:\gamma\in\mathcal{L}_{\pi_0}} \pi_{[\gamma]} \tag{5.41}$$

with $\pi_{[\gamma]} = \pi_{\omega\circ\gamma}$
We can now define, given an automorphism $\gamma \in \mathcal{L}_{\mathcal{C}}$, an isometry,$T_\gamma$ from $\mathcal{H}_0$ to $\mathcal{H}_{[\gamma]}$, by

$$\begin{aligned} T_\gamma\Omega &= \xi_\gamma \\ T_\gamma\gamma(A)\Omega &= \pi_{[\gamma]}(A)\xi_\gamma \text{ for all } A \in \mathfrak{A} \end{aligned} \tag{5.42}$$

where ξ_γ is the cyclic vector asociated with $\omega_0 \circ \gamma$. The operator T_γ intertwines the representations $(\pi_{[\gamma]}, \mathcal{H}_{[\gamma]})$ and $(\gamma, \mathcal{H}_0)$ of $\mathfrak{A}$

$$\pi_{[\gamma]}(A)T_\gamma = T_\gamma\gamma(A) \tag{5.43}$$

The definition of T_γ can be extended to arbitrary representations $(\pi_{[\hat{\gamma}]}, \mathcal{H}_{[\hat{\gamma}]})$,$\hat{\gamma} \in \mathcal{L}_{\pi_0}$

$$T_\gamma|_{\mathcal{H}_{[\hat{\gamma}]}} = T_{\hat{\gamma}\circ\gamma}T_{\hat{\gamma}}^{-1}|_{\mathcal{H}_{[\hat{\gamma}]}} \tag{5.44}$$

then

$$\pi_{[\hat{\gamma}]\circ\gamma}(A)T_\gamma|_{\mathcal{H}_{[\hat{\gamma}]}} = T_\gamma\pi_{\hat{\gamma}}(\gamma(A))|_{\mathcal{H}_{[\hat{\gamma}]}} \tag{5.45}$$

Definition 5.7 (Field Operators) *The field operators $\phi_\gamma(B)$, $B \in \mathcal{B}^{\mathcal{C}_a}$ are defined by*

$$\psi_\gamma(B)\Phi = T_\gamma\pi_{\hat{\gamma}}(B)\Phi \tag{5.46}$$

where Φ is an arbitrary vector in $\mathcal{H}_{[\hat{\gamma}]}$

Definition 5.8 (Localized Field) *A field $\phi_\gamma(B)$ is called localized in a space-like cone $\mathcal{C} \subseteq \mathcal{C}'_a$ iff there is a unitary operator $\Gamma \in \mathcal{B}^{\mathcal{C}_a}$ intertwining γ with $\hat{\gamma}$ such that $\hat{\gamma}$ is localized in $\mathcal{C}$ and $\Gamma B \in \mathcal{B}^{\mathcal{C}_a}$. The family of fields localized in $\mathcal{C}$ is called $\mathfrak{F}^{\mathcal{C}_a}(\mathcal{C})$. $\mathfrak{F}^{\mathcal{C}_a}(\mathcal{C})$ commutes with all operators in $\mathfrak{A}^c(\mathcal{C})$*

this definition is independent of the choice of the auxiliary cone $\mathcal{C}_a$ as a consequence of the following lemma

Lemma 5.2 *Let $\phi(B) \in \mathfrak{F}^{\mathcal{C}_a}(\mathcal{C})$ be a field localized in $\hat{\mathcal{C}}$ and $\gamma \in \mathcal{L}_{\mathcal{C}}$. If $\mathcal{S}$ is the minimal simple domain containing $\mathcal{C}$ and $\hat{\mathcal{C}}$ such that $\mathcal{S}$ is space-like to $\mathcal{C}_a$ and if $\hat{\mathcal{C}}_a$ is any other auxiliary cone space-like to $\mathcal{S}$ then $\phi_\gamma(B) \in \mathfrak{F}^{\hat{\mathcal{C}}_a}(\hat{\mathcal{C}})$, or*

$$\mathfrak{F}^{\mathcal{C}_a}(\mathcal{C}) = \mathfrak{F}^{\hat{\mathcal{C}}_a}(\hat{\mathcal{C}}) \tag{5.47}$$

In the following we drop the explicit reference π to the representation we work in, the chosen representation will be the representation on the Hilbert-space we are working on. In order to define our fields in a way we can define a statistics operator we have chosen apropriately localized representations

- Choose an auxiliary cone $\mathcal{C}_a$, with asymptotic $\alpha \in (0, 2\pi)$ and an opening angle $\epsilon \in (0, \pi)$.
- Choose a reference cone $\mathcal{C}_0$ with an asymptotic direction α_0 and an opening so small that $\mathcal{C}_0$ and $\mathcal{C}_a$ are spacelike separated.
- Let $\pi \in \mathcal{L}_{\pi_0}$ be a representation of $\mathfrak{A}$ localizable in cones relative to the vacuum representation π_0 and let $\gamma \in \mathcal{L}_{\mathcal{C}_0}$ be a representation equivalent to π and localized in $\mathcal{C}_0$.
- Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be space-like cones space-like to each other and with a union $\mathcal{C}_1 \cup \mathcal{C}_2$ space-like to $\mathcal{C}_a$
- Let $\gamma_1 \in \mathcal{L}_{\mathcal{C}_1}$ and $\gamma_2 \in \mathcal{L}_{\mathcal{C}_2}$ be representations unitary equivalent to γ.
- Let $\mathcal{S}_i \supseteq \mathcal{C}_0 \cup \mathcal{C}_i,\ i = 1, 2$ be a simple domains space-like to $\mathcal{C}_a$

Then we have unitary operators $\Gamma_i,\ i = 1, 2$ such that

$$\begin{aligned} \Gamma_i^* \gamma_i(A) \Gamma_i = \gamma(A), \text{ for all } A \in \mathfrak{A} \\ \Gamma_i \in \mathcal{B}(\mathcal{C}_0 \cup \mathcal{C}_i) \cap \overline{\mathfrak{A}(\mathcal{S}_i)}^w \end{aligned} \tag{5.48}$$

We define fields in the local field algebra $\mathfrak{F}^{\mathcal{C}_a}(\mathcal{C}_i)$ for $B_i \in \overline{\mathfrak{A}(\mathcal{C}_i)}^w$ by

$$\psi(\mathcal{C}_i, B_i) \equiv \psi(\Gamma_i^* B_i) = T_\gamma \Gamma_i^* B_i \tag{5.49}$$

the following identities hold for $\Phi \in \mathcal{H}_{[\hat{\gamma}]}$

$$\begin{aligned} \psi(\mathcal{C}_1, B_1)\psi(\mathcal{C}_2, B_2)\Phi &= T_{\gamma\circ\gamma} \gamma^{\mathcal{C}_a}(\Gamma_1^*) \Gamma_2^* B_2 B_1 \Phi \\ \psi(\mathcal{C}_2, B_2)\psi(\mathcal{C}_1, B_1)\Phi &= T_{\gamma\circ\gamma} \gamma^{\mathcal{C}_a}(\Gamma_2^*) \Gamma_1^* B_2 B_2 \Phi \end{aligned} \tag{5.50}$$

which leads to the following operator interchanging the two fields

Definition 5.9 (Statistics Matrix)

$$R_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2) = T_{\gamma\circ\gamma}\epsilon_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2)T_{\gamma\circ\gamma}^{-1} \tag{5.51}$$

where in the definition of the statistics matrix the statistics operator occurred which is defined by

$$\epsilon_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2) = \gamma^{\mathcal{C}_a}(\Gamma_1^*)\Gamma_2^*\Gamma_1\gamma^{\mathcal{C}_a}(\Gamma_2) \tag{5.52}$$

The statistics matrix $R_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2)$ and statistics operator $\epsilon_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2)$ describe the the commutation relations of space-like separated field operators.

The statistics operator ϵ satisfies the following theorem essential in understanding the consequences of topology

Theorem 5.3 *1. Suppose that $\hat{\mathcal{C}}_a$ is another auxilary cone space-like to the $\mathcal{S}_i$'s. Then*

$$\epsilon_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2) = \epsilon_{\gamma}^{\hat{\mathcal{C}}_a}(\gamma_1,\gamma_2). \tag{5.53}$$

2. Suppose that $\hat{\mathcal{C}}_i,\ i=1,2$ are space-like cones with $\hat{\mathcal{C}}_i \cup \mathcal{C}_i \subset \hat{\mathcal{S}}_i$ and the $\hat{\mathcal{S}}_i$'s mutualy space-like and space-like to $\mathcal{C}_a$ Let $\hat{\gamma}_i \in \mathcal{L}_{\hat{\mathcal{C}}_i}$ be unitary equivalent to γ. Then

$$\epsilon_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2) = \epsilon_{\gamma}^{\mathcal{C}_a}(\hat{\gamma_1},\hat{\gamma_2}). \tag{5.54}$$

3. The operator $\epsilon_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2)$ commutes with $\gamma^{\mathcal{C}_a}(\gamma^{\mathcal{C}_a}(\mathcal{B}^{\mathcal{C}_a}))$.

where the variations are restricted to those similar to the ones in figure [5.3] In

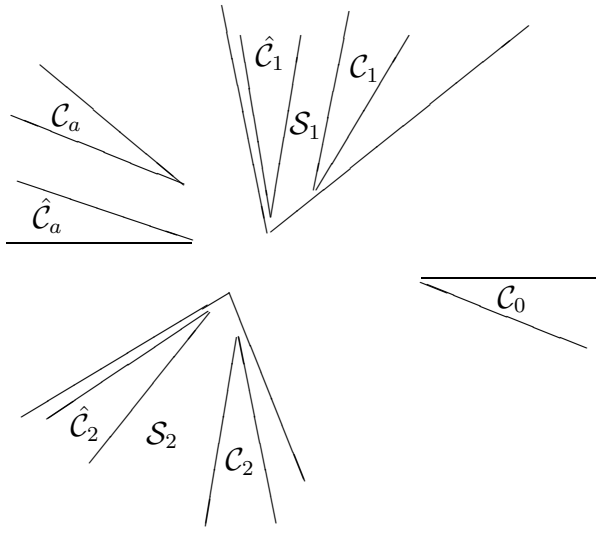


Figure 5.3: Variations leaving $\epsilon_{\gamma}^{\mathcal{C}_a}(\gamma_1,\gamma_2)$ invariant

the following picture the simple domains $\mathcal{S}_1$ containing $\hat{\mathcal{C}}_1 \cup \mathcal{C}_1$ also contain $\mathcal{C}_a$ or

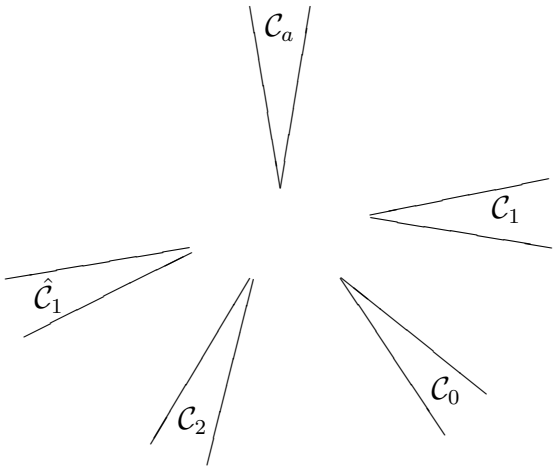


Figure 5.4: Variation not leaving $\epsilon_\gamma^{\mathcal{C}_a}(\gamma_1, \gamma_2)$ invariant

$\mathcal{C}_2$ and therefore the conditions in the definition [5.9] respectively theorem [5.3] are violated. Hence the statistics operator is not neccesarily invariant under these changes. We can without loss of generality fix the auxilary cone $\mathcal{C}_a$ and choose the asymptotic direction of $\mathcal{C}_a$ to be

$$\alpha_a = \pi \tag{5.55}$$

We define the asymptotic direction $as(\gamma)$ of an automorphism $\gamma \in \mathcal{L}$ to be the asymptotic direction of the space-like cone in which it is localized. And we require the asymptotic direction of γ to be in the following interval

$$-\pi + \frac{\epsilon}{2} \leq as(\gamma) \leq \pi - \frac{\epsilon}{2} \tag{5.56}$$

where ϵ is the opening angle of $\mathcal{C}_a$. If we now pick our $\gamma \equiv \pi \in \mathcal{L}_{\pi_0}, \gamma_1$ and γ_2 in accordance with demands formulated at the beginning of this section we can define

Definition 5.10

$$\begin{aligned} \epsilon_\gamma^{\mathcal{C}_a,>} &\equiv \epsilon_\gamma^{\mathcal{C}_a}(\gamma_1, \gamma_2) \text{ iff } as(\gamma_1) > as(\gamma_2) \\ \epsilon_\gamma^{\mathcal{C}_a,<} &\equiv \epsilon_\gamma^{\mathcal{C}_a}(\gamma_1, \gamma_2) \text{ iff } as(\gamma_2) < as(\gamma_1) \end{aligned} \tag{5.57}$$

we can even make the identification $\epsilon_\gamma^\# \equiv \epsilon_\gamma^{\mathcal{C}_a,\#}$ *as long as we satisfy the conditions in [5.3]. Now we can define*

$$R_\gamma^\# = T_{\gamma\circ\gamma}\epsilon_\gamma^\# T_{\gamma\circ\gamma}^{-1}, \text{ for } \# =>, < . \tag{5.58}$$

We can in a similar way define a statistics operator and a statistics matrix interchanging fields asociated with two non-equivalent representations. If we

define fields two fields

$$\begin{aligned}\psi_\gamma(\Gamma^* B_1) &= T_\gamma \Gamma^* B_1 \\ \psi_\gamma(\tilde{\Gamma}^* B_2) &= T_{\tilde{\gamma}} \tilde{\Gamma}^* B_2 \end{aligned} \tag{5.59}$$

where the unitary intertwiners $\Gamma, \tilde{\Gamma}$ satisfy

$$\begin{aligned}\gamma_1(A) &= \Gamma\gamma(A)\Gamma^* \\ \gamma_2(A) &= \tilde{\Gamma}\tilde{\gamma}(A)\tilde{\Gamma}^* \end{aligned} \tag{5.60}$$

for $A \in \mathfrak{A}, \gamma_i \in \mathcal{L}_{\mathcal{C}_i}$ and $B_i \in \overline{\mathfrak{A}(\mathcal{C}_i)}^w$ for $i = 1, 2$ and $\mathcal{C}_1$ space-like to $\mathcal{C}_2$. If furthermore we choose $\mathcal{C}$ and $\tilde{\mathcal{C}}$ such that $\mathcal{C} \cup \tilde{\mathcal{C}}$ is space-like to $\mathcal{C}_a$ we can redefine the statistics operator and the statistics matrix

$$\begin{aligned} R^{\mathcal{C}_a}_{\gamma,\tilde{\gamma}}(\gamma_1, \gamma_2) &= T_{\tilde{\gamma}\circ\gamma} \epsilon^{\mathcal{C}_a}_{\gamma,\tilde{\gamma}}(\gamma_1, \gamma_2)(T^{-1}_{\tilde{\gamma},\gamma\circ\gamma}) \\ \epsilon^{\mathcal{C}_a}_{\gamma,\tilde{\gamma}}(\gamma_1, \gamma_2) &= \tilde{\gamma}^{\mathcal{C}_a}(\Gamma^*)\tilde{\Gamma}^*\Gamma\gamma^{\mathcal{C}_a}(\tilde{\Gamma}) \end{aligned} \tag{5.61}$$

And for $as(\gamma_1) \overset{>}{<} as(\gamma_2)$ we define

$$\begin{aligned} \epsilon^{\overset{>}{<}}_{\gamma,\tilde{\gamma}} &\equiv \epsilon^{\mathcal{C}_a}_{\gamma,\tilde{\gamma}}(\gamma_1, \gamma_2) \\ R^{\overset{>}{<}}_{\gamma,\tilde{\gamma}}(\gamma_1, \gamma_2) &\equiv R^{\mathcal{C}_a}_{\gamma,\tilde{\gamma}}(\gamma_1, \gamma_2) \end{aligned} \tag{5.62}$$

and the following theorem holds

Theorem 5.4 *1. $\epsilon^{>}_{\gamma,\tilde{\gamma}}$ and $\epsilon^{<}_{\gamma,\tilde{\gamma}}$ are independent of $\mathcal{C}_a$.*

2. $\epsilon^{\overset{>}{<}}_{\gamma,\tilde{\gamma}}$ commute with $\gamma^{\mathcal{C}_a}(\tilde{\gamma}^{\mathcal{C}_a}(\mathcal{B}^{\mathcal{C}_a})) = \tilde{\gamma}^{\mathcal{C}_a}(\gamma^{\mathcal{C}_a}(\mathcal{B}^{\mathcal{C}_a}))$.

3. $\epsilon^{>}_{\gamma,\tilde{\gamma}}\epsilon^{<}_{\tilde{\gamma},\gamma} = \mathbb{1}$.

4. If γ is equivalent to γ' and $\tilde{\gamma}$ is equivalent to $\tilde{\gamma}'$ then

$$\epsilon^{\overset{>}{<}}_{\gamma',\tilde{\gamma}'} = U^* \epsilon^{\overset{>}{<}}_{\gamma,\tilde{\gamma}} U \tag{5.63}$$

for some unitary operator $U \in \mathcal{B}^{\mathcal{C}_a}$ only depending on $\gamma, \tilde{\gamma}, \gamma', \tilde{\gamma}'$

5. $\psi_\gamma(\Gamma^* B_1)\psi_{\tilde{\gamma}}(\tilde{\Gamma}^* B_2) = R^{\overset{>}{<}}_{\gamma,\tilde{\gamma}}\psi_{\tilde{\gamma}}(\tilde{\Gamma}^* B_2)\psi_\gamma(\Gamma^* B_1)$, for $as(\mathcal{C}_1) \overset{>}{<} as(\mathcal{C}_2)$

6. With the obvious meaning of $\Gamma, \hat{\Gamma}, \mathcal{C}_a$ and $\hat{\mathcal{C}}_a$ we have

$$\epsilon^{<}_{\gamma,\tilde{\gamma}} = \tilde{\gamma}^{\mathcal{C}_a}(\Gamma^*)\tilde{\gamma}^{\mathcal{C}_a}(\hat{\Gamma})\epsilon^{>}_{\gamma,\tilde{\gamma}}\hat{\Gamma}^*\Gamma. \tag{5.64}$$

The distinction made in this theorem, especially in point five, between the $R^<, \epsilon^<$ and the $R^>, \epsilon^>$ and necessitated by theorem [5.3] reflects the influence of topology on the statistics operator and the statistics matrix. It is this distinction that leads to braid statistics in 2+1 -dimensions for fields localized in unbounded regions of space-time.

5.4 Braid group representation

In this section we present a theorem that shows that the fields obtained in the preceding part of this chapter carry representations of the braid group.
The groupoid B_n^c on n coloured strands is obtained from B_n by assigning different colours to all strands and requiring that two n-coloured braid diagrams be composable only if the colours of the strands match. Where a groupoid is a mathematical object equal to a group except that the composition law is restricted to specified combinations of elements, where it has to be understood that the existence of the composition $a \circ b$ does not imply the existence of the composition $b \circ a$.
If $\dot{\gamma}_1, \ldots, \dot{\gamma}_n$ be n morphisms of the algebra $\mathcal{B}^{\mathcal{C}_a}$ localized in space-like cones $\dot{\mathcal{C}}_1, \ldots, \dot{\mathcal{C}}_2$, with the property that $\dot{\mathcal{C}}_i$ is space-like to $\dot{\mathcal{C}}_j$ if $i \neq j$ and $\dot{\mathcal{C}}_i$ is space-like to $\mathcal{C}_a$ for some auxilary cone $\mathcal{C}_a$, $i = 1, \ldots, n$. Let Γ_i be unitary intertwiners such that

$$\gamma_i(A) = \Gamma_i \dot{\gamma}_i(A) \Gamma_i^*, \text{ for } A \in \mathfrak{A}, i = 1, \ldots, n \tag{5.65}$$

and such that that γ_i is localized in $\mathcal{C}_i$, with $\mathcal{C}_i$ space-like to $\mathcal{C}_j$, for $i \neq j$ and $\mathcal{C}_i$ space-like to $\mathcal{C}_a$. And we define

$$\psi_i(\mathcal{C}_i) = T_{\dot{\gamma}_i} \Gamma_i^* B_i \tag{5.66}$$

with $B_i \in \overline{\mathfrak{A}\mathcal{C}_i}^w$. We can interchanging $\psi_i(\mathcal{C}_i)$ and $\psi_{i+1}(\mathcal{C}_{i+1})$ in the product $\prod_{i=1,\ldots,n} \psi_i(\mathcal{C}_i)$ where ψ_i stands left from ψ_{i+1} by applying a general matrix operator

$$R_i^{\overset{<}{>}}(\dot{\gamma}_1, \ldots, \dot{\gamma}_n) = T_{\dot{\gamma}_1} \ldots, T_{\dot{\gamma}_{i+1}} \epsilon_{\dot{\gamma}_i \dot{\gamma}_{i+1}} T_{\dot{\gamma}_{i+1}}^{-1} T_{\dot{\gamma}_i}^{-1} \tag{5.67}$$

Let $b \in B_n$ represent a word in the generators $\{\tau_1^{\pm 1}, \ldots, \tau \pm 1_n\}$ that is

$$b = \tau_{i_k}^{\epsilon_k} \ldots \tau_{i_1}^{\epsilon_1} \tag{5.68}$$

for some $k \in \mathbb{N}$ and where $\epsilon_j = \pm 1, i_j \in 1, \ldots, n-1$, for all $j = 1, \ldots, k$. I we further define $b_m = \tau_{i_m}^{\epsilon_m} \ldots \tau_{i_m}^{\epsilon_m}$, and we define π_m to be the permutation of the endpoints of n-coloured strands corresponding to the element $b_m \in B_N$, with $\tau_0 \equiv 1$ then the following theorem holds

Theorem 5.5 *If we define*

$$R(\tau_i^{\pm 1}) \equiv R^{\overset{>}{<}}(\dot{\gamma}_{\pi(1)}, \ldots, \dot{\gamma}_{\pi(n)}) \tag{5.69}$$

If we choose an arbitrary localized automorphism, $[\hat{\gamma}] \in \mathcal{L}_{\pi_+ 0}$, and we put

$$\mathcal{H} \equiv \mathcal{H}_{[\hat{\gamma} \circ \dot{\gamma}_n \ldots \circ \dot{\gamma}_1]} \tag{5.70}$$

and if furthermore we put

$$R(b; \dot{\gamma}_1, \ldots, \dot{\gamma}_n) = \prod_{m=k}^{1} R(\tau_{i_m}^{\epsilon_m}, \pi_{m-1}) \tag{5.71}$$

The map

$$B_n \ni b \rightarrow R(b; \dot{\gamma}_1, \ldots, \dot{\gamma}_n) \tag{5.72}$$

defines a unitary representation of the groupoid ,B_n^c of n-coloured strands on the Hilbert-space$\mathcal{H}$

Chapter 6

Discussion

In this report we give account of an attempt to construct a solid connection between the phenomenology of the Fractional Quantum Hall-effect and the duality axiom used in $(2+1)$-dimensional Algebraic Quantum Field Theory.
We found that there are two obstacles in establishing this connection. The first one is the connection between the phenomenological theories of the incompressible quantum fluid originally formulated by Laughlin and $U(1) \times SU(2)$ gauge theory satisfying strong clusterproperties as studied by Fröhlich and Studer [FS92]. The second is the connection between $U(1) \times SU(2)$ gauge theory in the scaling limit and Algebraic Quantum Field theory.
A solution to the second obstacle has been pointed out by Fröhlich, Gabbiani and Marchetti [FGM], and is based on an observation of Hasenfratz [Has79] in the context of Higgs-models with a Chern-Simons term.
We found a direction to search for a solution to the first obstacle in an Theorem by Borchers [Bor62], we were able to translate this theorem to (2+1)-dimensions. The theorem however is applicable only to Bosons, this is a serious problem when studying quasi-particles with fractional statistics. Up to now we were unable to generalize the theorem to (quasi-)particles satisfying fractional statistics. However it is possible that we don't need such a general theorem but that we can restrict our selves to reconstructing the theorem for "even" fields that is products of fields satisfying Bose-statistics.
Resuming we see the first obstacle still unsolved, but we think we have made progress by pointing out a direction which in the future might give the answer.

Appendix A

A.1 Some Basic Concepts of Topology and Geometry

In this appendix we sketch the topological and geometrical concepts we used in this report, a more complete treatment can be found in any good book on topology and geometry. An introductory text I liked using was the book "Topology and Geometry for Physicist" by Charles Nash and Siddhartha Sen [NS83], a more comprehensive text is "Geometry, Topology and Physics" by M. Nakahara [Nak90]. Topology and Geometry are related because some geometrical objects are related to topological invariants, an example is the Gauss-Bonnet Theorem which connects the geometric notion of curvature to a topological invariant the Euler-Poincaré characteristic. In the context of the Quantum Hall-Effect the conductivity is believed to be a topological invariant.
The basic topological structure in geometry is the manifold, a manifold is topological space $\mathbb{M}$ provided with an additional structure of so called charts. A chart is a pair constituted by a open subset $\mathbb{M}_\alpha \in \mathbb{M}$ and a map ϕ_α. The map ϕ_α is a homeomorphism from $\mathbb{M}_\alpha$ into an open subset of $\mathbb{R}^n$. There is a set of $(\mathbb{M}_\alpha, \phi_\alpha)$ charts defined on the manifold such that the following holds

- The $\mathbb{M}_\alpha$ cover $\mathbb{M}$: $\bigcup_\alpha \mathbb{M}_\alpha = \mathbb{M}$
- If two of the $\mathbb{M}_\alpha$ overlap $\mathbb{M}_{\alpha_1} \cap \mathbb{M}_{\alpha_2} \neq \emptyset$ then the map $\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1}$ from $\phi_{\alpha_1}(\mathbb{M}_{\alpha_1} \cup \mathbb{M}_{\alpha_2})$ to $\phi_{\alpha_2}(\mathbb{M}_{\alpha_1} \cup \mathbb{M}_{\alpha_2})$ is infinitely differentiable.

The family of charts is called an atlas, which refers to a well known example of a topological space the sphere which can be asociated with the globe.
In any point on the manifold $\mathbb{M}$ we can define a tangent space by the regular definition of tangent space $T_x(\mathbb{M})$ on $\mathbb{R}^n$. That is with every point $x \in \mathbb{M}$ we can asociate an vector space which is spanned by ∂_i where ∂_i denotes differentiation with respect to the coordinates in a $\phi_\alpha(\mathbb{M}_\alpha \ni x)$. To see how to implement this tangent space to describe a contravariant vector field we look how it can be used to describe a flow in the plane, if we parametrize the trajectory through $\Psi(t = 0, x) = x$ in a flow Ψ by t then we have the following velocity field

$$v_i(x) = \partial_t \Psi_i(t = 0, x) \tag{A.1}$$

and we can describe the rate of change of a function $f(\Psi(t,x))$ along the flow by

$$\begin{aligned}\frac{d}{dt}f(\Psi(0,x)) &= \sum_i \partial_{x_i}\frac{d}{dt}\Psi_i(0,x) \\ &= \sum_i v_i(x)\partial_i f(x) \end{aligned} \tag{A.2}$$

where the components of the velocity have become coefficients of a vector in the tangent bundle, a contravariant vector. And if we set $f(x) = x_i$ then we get

$$\sum_j v_j(x)\partial_j x_i = \sum_j \delta_{ij}v_j(x) = v_j(x) \tag{A.3}$$

which is one component of the velocity vector.
To introduce covariant vector we need to define the the cotangent space $T_x^*(\mathbb{M})$, the cotangent space at a point x is the space of linear maps from $T_x(\mathbb{M})$ to $\mathbb{R}$. An example of an element of the cotangent space is the differential of a function

$$df = \frac{\partial f}{\partial x^i}dx^i \tag{A.4}$$

and it defines a linear operator on $T_x(\mathbb{M})$ by

$$\langle df, \sum_i v_i(x)\partial_{x^i}\rangle \equiv \sum_i v_i(x)\partial_{x^i}f \tag{A.5}$$

We can define 1-forms $\boldsymbol{\omega} \in T_x^*(\mathbb{M})$ by

$$\boldsymbol{\omega} \equiv a_i dx^i \tag{A.6}$$

where the simultaneous occurrence of an upper and an lower index i denotes summation over this index. Furthermore we have an orthogonality relation

$$\langle dx^i, \partial_{x^j}\rangle = \delta^i_j \tag{A.7}$$

We can generalize from (co)tangent space to tensor spaces by defining

$$\mathbf{T}^a_b \equiv \bigotimes_{i=1}^{i=a} T_x(\mathbb{M}) \otimes \bigotimes_{j=1}^{j=b} T_x^*(\mathbb{M}) \tag{A.8}$$

A simple element of this tensor space can be denoted by by

$$T^{i_1,\ldots,i_a}_{j_1,\ldots,j_b} dx^{i_1}\ldots dx^{i_a} dx_{j_1}\ldots dx_{j_b} \tag{A.9}$$

where we have reexpressed ∂_{x^i} in the common way as dx_i. A n-form can be defined in the following way by antisymmetrizing elements of $\mathbf{T}^n_0$

$$dx^{i_1}\wedge\ldots\wedge dx^{i_n} = \sum_\sigma \mathrm{sgn}(\sigma) dx^{\sigma(i_1)}\otimes\ldots\otimes dx^{\sigma(i_n)} \tag{A.10}$$

where the sum over σ denotes a sum over all permutations of the indices $i_1, \ldots, i_n$ and $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation.
Now a more general definition of the exterior derivative d we used before can be given for n-forms $\boldsymbol{\omega} = \omega_{i_1 \ldots i_n}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_n}$

$$d\boldsymbol{\omega} \equiv (\partial_{x^{i_{n+1}}} \omega_{i_1 \ldots i_n}(x)) dx^{i_{n+1}} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_n} \tag{A.11}$$

the exterior derivative has the following properties

- $d^2 = 0$
- If $\boldsymbol{\omega}$ is an n-form on an n-dimensional manifold then $d\boldsymbol{\omega} = 0$
- Stokes Theorem: If $\mathbb{M}$ is a n-dimensional manifold and $\partial\mathbb{M}$ its boundary and $\boldsymbol{\omega}$ is a k-form with $0 \leq k < n$ then the following holds

$$\int_{\mathbb{M}} d\boldsymbol{\omega} = \int_{\partial\mathbb{M}} \boldsymbol{\omega} \tag{A.12}$$

The last statement is a bit premature because we have not defined integration on manifolds yet, now we come to that. We can define a measure which can be given in one coordinate patch on an n-dimensional manifold by an n-form

$$\boldsymbol{\omega} = dx^1 \wedge \ldots \wedge dx^n \tag{A.13}$$

this measure behaves like a volume element under transformation of the coordinates and can be asociated with ordinary integration of a function f over the coordinate patch in the following way

$$\int_{\Phi(\mathbb{M}_\alpha)} f\boldsymbol{\omega} = \int_{\Phi(M_\alpha)} f dx^1 dx^2 \ldots dx^n \tag{A.14}$$

to define the integral over the whole manifold we need a partition of unity, where we will use the notation $U_\alpha \equiv \phi_\alpha(\mathbb{M}_\alpha)$ to denote a coordinate patch.

Definition A.1 (Partition of Unity) *A partition of unity is a set of differentiable functions $e_\alpha(x)$ asociated with an open covering $\{U_\alpha\}$ of $\mathbb{M}$ which is locally finite,that is only a finite number of U_α contain a point x,with the following properties*

- $0 < e_\alpha(x) < 1$
- $e_\alpha(x) = 0$ *if* $x \notin U_\alpha$
- $e_1(x) + e_2(x) + \ldots = 1$

The partition of unity can be used to define the integral over whole $\mathbb{M}$

$$\int_{\mathbb{M}} f\boldsymbol{\omega} \equiv \sum_\alpha \int_{U_\alpha} e_\alpha(x)(x^1_\alpha, \ldots, e^n_\alpha) f(x^1_\alpha, \ldots, x^n_\alpha) dx^1_\alpha \ldots dx^n_\alpha \tag{A.15}$$

this definition of the integral is independent of the chosen partition of unity.

A.2 Fibrations

On a manifold we can construct a fibration, in fact we can construct it on any topological space but we restrict ourselves here to manifolds.
Locally we can define a produkt space of the manifold with another topological space called the fibre F such that we have a topological space E which is locally $U_\alpha \times F$. Furthermore there is a group G of homeomorphism of the fibre F and with every coordinate patch an homeomorphism is given

$$\phi_\alpha : \Pi^{-1}(U_\alpha) \to U_\alpha \times F \tag{A.16}$$

with the following property

$$\Pi\phi_\alpha^{-1}(x, f) = x \tag{A.17}$$

where $x \in U_\alpha, f \in F$.
The group G is used in describing how to take the fibre from one coordinate patch to an other, that is the transition functions $g_{\alpha\beta} \equiv \phi_\alpha \circ \phi_\beta^{-1}$ describing the transition from the fibre in a coordinate patch U_β to the fibre asociated with a coordinate patch U_α take their values in G. The transition functions, together with the local finite covering and the fibre give a complete description of the total space E and by the following algorythm we are able to construct E. We start with constructing the union of a direct product space asociated with the open sets U_α

$$\tilde{E} = \bigcup_\alpha U_\alpha \times F \tag{A.18}$$

then we define E to be the set of equivalence classes under the following identification of $(x, f) \in U_\alpha \times F$ with $(x', f') \in U_\beta \times F$

$$\begin{aligned} (x, f) \quad &\sim \quad (x', f') \Leftrightarrow \\ x = x' \quad &\text{and} \quad g_{\alpha\beta}(x) f = f' \end{aligned} \tag{A.19}$$

and we can define the projection Π by

$$\Pi[(x, f)] = x \tag{A.20}$$

where the square brackets denote the equivalence class under the equivalence relation before defined. We can define ϕ_α by

$$\phi_\alpha^{-1}(x, f) = [(x, f)] \tag{A.21}$$

Now we will proceed and make a big step to structures in which the structure group and base space are differentiable structures. That is we will assume the structure group to be a Lie group and we assume the base space to be a differentiable manifold. Now we would like to differentiate a continuous map from the manifold into the fibre in order to be able to do this we need to be able to compare the fibre at a point $x \in \mathbb{M}$ with the fibre in a point at an infinitesimal

distance from x. This however is not trivial and in order to do this we need to define a connection.
We now give such a connection 1-form and show how it can be used to uniquely define parallel transport of an element of the fibre along a trajectory γ. A connection one form is of the following form

$$\mathbf{A} \equiv A^a_\mu(x)(\frac{\lambda_a}{2i})dx^\mu \tag{A.22}$$

and the λ_a satisfies the folowing commutation relations

$$\left[\frac{\lambda_a}{2i}, \frac{\lambda_b}{2i}\right] = f_{abc}\frac{\lambda_c}{2i} \tag{A.23}$$

where f_{abc} are the structure constants of the Lie- algebra $\mathfrak{g}$. This connection one form is used to define an other 1-form which uniquely defines parallel transport

$$\boldsymbol{\omega} = g^{-1}dg + g^{-1}\mathbf{A}g \tag{A.24}$$

we now show how this Lie-algebra valued 1-form does define uniquely parallel transport, however we have to keep in mind that the choice of the 1-form $\mathbf{A}$ contains some arbitrariness.
The connection 1-form $\boldsymbol{\omega} \in \mathfrak{g} \times T^*_u P$ is a projection of the tangent space $T_u P$ of the principal bundle onto the vertical subspace $V_u P$, the vertical subspace $V_u P$ is the vector space tangent to the fibre G at the point $u = (x, g) \in P$ in the principal bundle and $V_u P \sim \mathfrak{g}$.
Now the horizontal subspace can be defined by

$$H_u P \equiv \{X \in T_u P | \boldsymbol{\omega}(X) = 0\} \tag{A.25}$$

and this leads to the following definition connecting a value in the fibre at a point $\gamma(0) \in \mathbb{M}$ to an equivalent value at an point $\gamma(t) \in \mathbb{M}$

Definition A.2 (Horizontal Lift) *Let $P(\mathbb{M}, G)$ be a G-bundle and let $\gamma : [0,1] \to \mathbb{M}$ be a curve in $\mathbb{M}$. A curve $\tilde{\gamma} : [0,1] \to P$ is said to be a horizontal lift of γ if $\Pi\tilde{\gamma} = \gamma$ and the tangent vector to $\tilde{\gamma}(t)$ always belongs to $H_{\tilde{\gamma}(t)}P$*

That this definition does the job is a consequence of the following theorem

Theorem A.1 (Unique Horizontal Lift) *Let $\gamma : [0,1] \to \mathbb{M}$ be a curve in $\mathbb{M}$ and let $u_0 \in \Pi^{-1}(\gamma(0))$. Then there exists a unique* **horizontal lift** *$\tilde{\gamma}(t)$ in P such that $\tilde{\gamma}(0) = u_0$.*

this theorem assigns to every $\gamma(t) \in \mathbb{M}$ a unique $\tilde{\gamma}(t) \in P$. Because every tangent vector $\tilde{X}$ of $\tilde{\gamma}$ satisfies $\boldsymbol{\omega}(\tilde{X}) = 0$, $\tilde{\gamma}$ is the solution of an ordinary differential equation and therefore it is unique. Now we can define the covariant derivative, which is nothing more than the rate of change with respect to t along the horizontal lift $\tilde{\gamma}(t)$

Definition A.3 (Covariant Derivative) *The covariant derivative is given as a function of the connection 1-form by*

$$D_\mu \equiv \frac{\partial}{\partial x^\mu} - \mathbf{A}^a_\mu R_a \tag{A.26}$$

where $R_a = (\lambda^a/2i)g(\partial/\partial g)$.

and there is the connection 2-form or curvature form $\mathbf{F}$ describing the commutation properties of the covariant derivatives

$$[D_\mu, D_\nu] = -F^a_{\mu\nu} R_a \tag{A.27}$$

and

$$\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} \tag{A.28}$$

A.3 Topological Invariants

A topological invariant is an object defined on a topological space which is invariant under a certain class of transformations of the topological space, homeomorpisms. A homeomorphism α is a continuous, invertible map from a space X to a space Y.

$$\begin{aligned} \alpha : X &\to Y \\ Y &= \alpha(X) \\ X &= \alpha^{-1}(Y) \end{aligned} \tag{A.29}$$

X and Y are called homeomorphic if such a map α exist. If such a map exist then it is an equivalence relation. If X is homeomorphic to Y and Z then we can compose the asociated homeomorphism forming a homeomorphism between Y and Z

$$\begin{aligned} \alpha_Y(X) &= Y \\ \alpha_Z(X) &= Z \\ &\Rightarrow \\ \beta(Y) \equiv \alpha_Z(\alpha_Y^{-1}(Y)) &= Z \end{aligned} \tag{A.30}$$

and β is also continuous and invertible.
Now we give three examples of topological invariants

- $dim(X)$ the dimension of a topological space X is a topological invariant
- $\pi_1(X)$ the fundamental group is an topological invariant
- The Euler-Poincaré characteristic $\chi(X)$ is a topological invariant

The fundamental group is the group of equivalence classes of embeddings of the circle in the topological space X. Two embeddings of the circle in a topological space are called equivalent if they can be transformed into each other by an continuous transformation. To be more precise to embeddings γ_i of the circle in X, which are in essence just closed paths in X,

$$\gamma_i : S^1 \to X \tag{A.31}$$

are equivalent if there is a continuous map $\gamma(t)$ from an interval $[0, 1]$ into the embeddings satisfying

$$\begin{aligned} \gamma(0) &= \gamma_1 \\ \gamma(1) &= \gamma_2 \end{aligned} \tag{A.32}$$

a simple example of π_1 is given by the winding number around a singularityin the theory of analytic functions .
The Euler-Poincaré characteristic is an more complicated object and we will give here its definition for some simple examples of 2-dimensional manifolds. For calculating the Euler-Poincaré characteristic we need a so called triangulation of the two dimensional manifold, that is we build up a surface out of triangles such that it can be continuously deformed into the manifold we are interested in. As examples we give the triangulations of the sphere and the torus The three dimensional structure is put in the numbering of the vertices, the so called 0-simplexes. If two or more vertices have the same number they represent the same point, 0-simplex. A line joining two points is called a 1-simplex and if two or more lines have the same pair of numbers at their end points they represent the same 1-simplex. A 2-simplex is a triangle, and two 2-simplexes are equivalent if they have the same 0-simplexes at their corners. Now we are able to calculate the Euler-Poincaré characteristic $\chi(\mathbb{M})$ by merely counting the number of 0,1 and 2-simplexes and adding their numbers in the appropriate way

$$\chi(\mathbb{M}) \equiv \sum_{p=0}^{2} (-1)^p n(p, \mathrm{tri}(\mathbb{M})) \tag{A.33}$$

where $n(p, \mathrm{tri}(\mathbb{M}))$ denotes the number of p-simplexes in a triangulation $\mathrm{tri}(\mathbb{M})$ of the two manifold $\mathbb{M}$ and $p = 0, 1, 2$.
For the triangulation of the sphere we have the following set of simplexes

- 4 0-simplexes: 1,2,3,4
- 6 1-simplexes: 12,13,14,23,24,34
- 4 2-simplexes: 123,124,234,134

thus leading to a Euler-Poincaré characteristic of $\chi(S^2) = 2$, in a similar way we find for the torus $\chi(\mathbb{T}) = 7$.

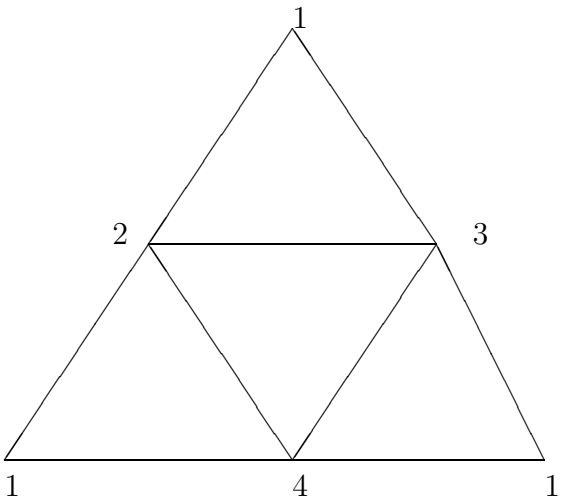


Figure A.1: triangulation of a sphere

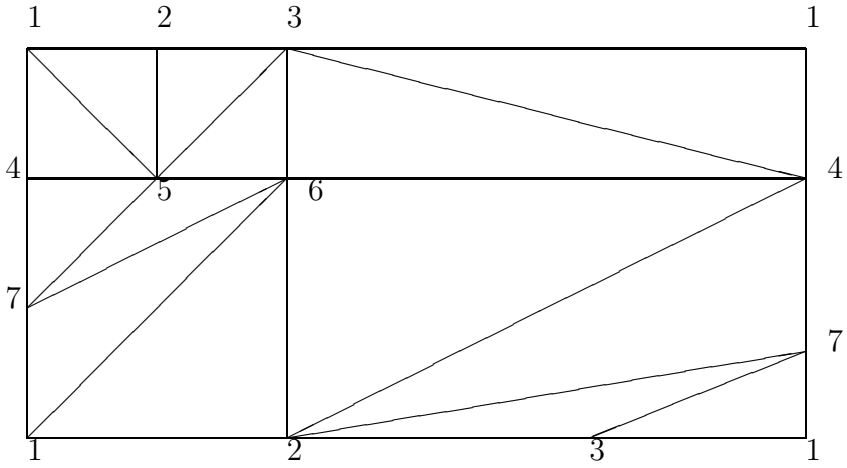
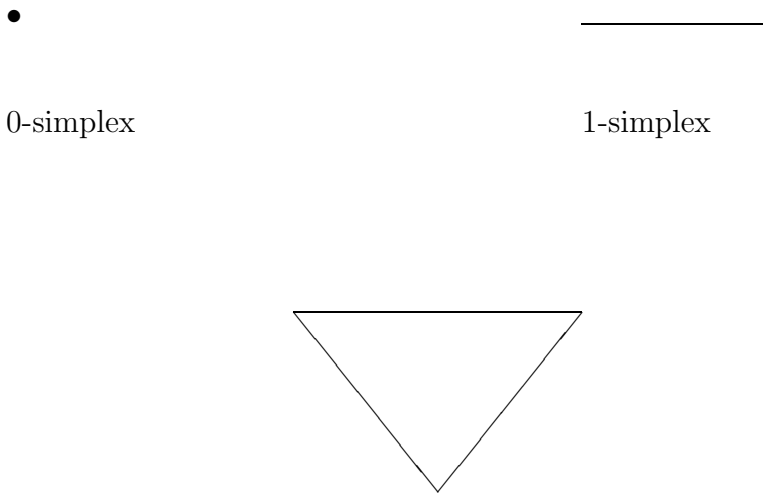


Figure A.2: triangulation of a torus

A.4 Quantized Hall-conductance as an topological invariant

A.4.1 Hall-conductance

In a paper by Qian Niu, D.J. Thouless and Yong-Shi Wu the conductivity was shown to be a topological invariant. They proved that the conductivity is



the first Chern class of a $U(1)$ principal fiber bundle of the groundstate wave functions on the base manifold of a torus $\mathbf{T}^2$. They took the Kubo-formula as a starting point for their analysis, the Kubo formula applies to a system of electrons in a uniform electric field satisfying periodic boundary conditions [JM73] (p.747-p.749). Here we merely state the results of their work, details can be found in their original paper [NTW85] and in [Koh85]. Furthermore the underlying theorems which account for the above statement are sketched. The Kubo-formula is given by:

$$\sigma = \frac{\imath e^2 \hbar}{A} \sum_{n>0} \frac{(v_1)_{0n}(v_2)_{n0} - (v_2)_{0n}(v_1)_{n0}}{(E_o - E_n)^2} \tag{A.34}$$

where $A = L_1 L_2$ the area of the system. The v_i are velocity operators and the subscripts denote the states between which the operators are sandwiched. That is

$$(v_i)_{kl} = \left\langle k \left| \sum_{j=1}^{N} \frac{1}{m_j} \left(-\imath\hbar \frac{\partial}{\partial x_{i,j}} + A_i \right) \right| l \right\rangle \tag{A.35}$$

here $\langle 0|$ denotes the groundstate of the system and $\langle n|$ an exited state. The summation over j is a summation over all electrons present and the subscript i denotes the direction. Using gauge-covariant boundary conditions and rewriting the expression interms of derivatives of the Hamiltonian Niu et al. obtain

$$\sigma = \frac{e^2}{h} \int_0^{2\pi} \int_0^{2\pi} d\vartheta d\varphi \frac{1}{2\pi\imath} \left[\left\langle \frac{\partial \phi_0}{\partial \vartheta} \mid \frac{\partial \phi_0}{\partial \varphi} \right\rangle - \left\langle \frac{\partial \phi_0}{\partial \varphi} \mid \frac{\partial \phi_0}{\partial \vartheta} \right\rangle \right] \tag{A.36}$$

which is equivalent to a constant times the first Chern-character. Thus the Hall-conductance is a topological invariant.

A.4.2 First Chern-class

The first Chern class can be calculated as soon as the curvature two-form is known. The underlying mathematics can largely be neglected making use of the following two theorems

Theorem A.2 *If $\mathbb{F}$ is a curvature 2-form on a bundle. Then the polynomials $P_j(\mathbb{F})$ in the expansion of the determinant*

$$\mathrm{Det}(t\mathbb{1} + \frac{\imath\mathbb{F}}{2\pi}) = \sum_J t^j P_{m-j}(\mathbb{F}) \tag{A.37}$$

are the j-th Chern class $c_j(P)$ of the bundle.

The second theorem is the generalized Gauss-Bonnet theorem linking the first Chern-character to a topological invariant.

Theorem A.3 *If $\mathbb{M}$ is a compact orientable manifold, $c_j(\mathbb{F})$ the j-th Chern class and $\chi(\mathbb{M})$ is the Euler-Poincaré characteristic then the following equality holds*

$$\int_{\mathbb{M}} c_j(\mathbb{F}) = \chi(\mathbb{M}) \tag{A.38}$$

Now these theorems can be related to the Quantum Hall-effect by taking $\mathbb{M} = \mathbb{T}^2$ the two torus and Berry's connection for the connection 1-form

$$A_{Berry} = \langle\phi_0|\partial_\vartheta\phi_0\rangle\, d\vartheta + \langle\phi_0|\partial_\varphi\phi_0\rangle\, d\phi \tag{A.39}$$

leading to Berry's curvature 2-form

$$\mathbb{F} = dA = (\langle\partial_\varphi\phi_0|\partial_\vartheta\phi_0\rangle\langle\partial_\vartheta\phi_0|\partial_\varphi\phi_0\rangle)\, d\varphi \wedge d\vartheta \tag{A.40}$$

Putting things together we obtain

$$\int_{\mathbb{T}^2} \frac{\imath\mathbb{F}}{2\pi} = \chi(\mathbb{T}^2) \tag{A.41}$$

The left hand side is up to a constant exactly the right hand side of equation [A.36] and thus the conductance is a topological invariant.

Appendix B

B.1 Basic concepts of C^*-algebra's

In chapter [5] we made extensive use of C^*-algebra's, in this appendix we give the basic theory behind C^*-algebra's.

Definition B.1 *An algebra $\mathcal{A}$ is a set with the following relations between the elements*

- *If $A, B \in \mathcal{A}$ then $A + B \in \mathcal{A}$*
- *If $A, B, C \in \mathcal{A}$ then there is a product defined such that $AB \in \mathcal{A}$ and $A(BC) = (AB)C = ABC$*
- *If $A, B, C \in \mathcal{A}$ then $(B + C) = AB + AC$*
- *If $A \in \mathcal{A}$ and $c \in \mathbb{C}$ then $cA \in \mathcal{A}$*

Definition B.2 *An algebra $\mathcal{A}$ is called a *-algebra if there is an involution * such that*

- $A^{**} = A$
- $(cA)^* = \overline{c}A^*$
- $(A + B)^* = A^* + B^*$
- $(AB)^* = B^*A^*$

If an algebra is equipped with a norm $\|\cdot\|$ and is furthermore closed with respect to this norm it is called a Banach-algebra.

Definition B.3 (C^*-algebra) *A C^*-algebra is an algebra $\mathfrak{A}$ satisfying the following requirements*

- *$\mathfrak{A}$ is a Banach algebra*
- *$\mathfrak{A}$ is a *-algebra*

- $\|A^*A\| = \|A\|^2$ *for* $\forall A \in \mathfrak{A}$

The definition of C^*-algebra's is motivated by the properties of the set of bounded linear operators on a Hilbert-space $\mathcal{H}$ equiped with the norm topology

$$\|A\| \equiv \sup_{\Psi \in \mathcal{H}} \frac{\|A\Psi\|_{\mathcal{H}}}{\|\Psi\|_{\mathcal{H}}} \tag{B.1}$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the norm on the Hilbert-space $\mathcal{H}$ asociated with the scalar product $\langle\cdot|\cdot\rangle$ defined on $\mathcal{H}$. A state in quantum mechanics is a normalized positive linear functional, mapping the observable algebra into the real numbers, thus giving the expectation value for the observable in the given state. In Algebraic Quantum Field Theory this is used to define states on the observable algebra.

Definition B.4 (State) *A state ω is defined by the following properties*

- ω *is a bounded linear functional on* $\mathfrak{A}$

$$\omega(\alpha A + \beta B) = \alpha\omega(A) + \beta\omega(B) \text{ for } A, B \in \mathfrak{A} \tag{B.2}$$

- ω *is normalized*

$$\|\omega\| = 1 \tag{B.3}$$

  where the norm on the form is given by the operator norm

$$\|\omega\| \equiv \sup_{A \in \mathfrak{A}} \frac{|\omega(A)|}{\|A\|} \tag{B.4}$$

- ω *is positive*

$$\omega(A^*A) \leq 0 \tag{B.5}$$

Given the C^*-algebra $\mathfrak{A}$ of observables and a state ω defined on it it is possible to construct a Hilbert space $\mathcal{H}_\omega$ carrying a representation π_ω of the observable algebra. Where a representation is a map respecting the algebraic structure, i.e.

$$\begin{aligned} \pi_\omega(AB) &= \pi_\omega(A)\pi_\omega(B) \text{ for } A, B \in \mathfrak{A} \\ \pi_\omega(\lambda A) &= \lambda\pi_\omega(A) \text{ for } A \in \mathfrak{A}, \lambda \in \mathbb{C} \end{aligned} \tag{B.6}$$

Furthermore it can be shown that there exists a cyclic vector if the observable algebra contains a unit element. We will now in short sketch the construction of the Hilbert space. First use the state ω to define a scalar product on $\mathfrak{A}$

$$\langle A|B\rangle = \omega(A^*B) \tag{B.7}$$

where $A, B \in \mathfrak{A}$. The scalar product is used to define a subset $\mathfrak{J}$ of $\mathfrak{A}$ by

$$\mathfrak{J} = \{X \in \mathfrak{A} | \langle X|X\rangle = 0\} \tag{B.8}$$

now this subset is used to define an equivalence relation on $\mathfrak{A}$

$$A \equiv B \text{ if } A + X = B \text{ and } X \in \mathfrak{J} \tag{B.9}$$

the equivalence classes defined by the equivalence relation form a pre-Hilbert space $\mathfrak{A}/\mathfrak{J}$, because the scalar product defined by ω on $\mathfrak{A}$ can be used to define a scalar product on the equivalence classes. Taking the norm closure of $\mathfrak{A}/\mathfrak{J}$ we obtain the Hilbert space $\mathcal{H}_\omega$

$$\mathcal{H}_\omega = \overline{\mathfrak{A}/\mathfrak{J}}^n \tag{B.10}$$

A representation of the observable algebra on the pre-Hilbertspace $\mathfrak{A}/\mathfrak{J}$ can now be defined by

$$\pi_\omega(A)\Psi = [AB] \text{ for } \Psi = [B] \tag{B.11}$$

If $\mathfrak{A}$ is a C^*algebra π_ω can be extended to a representation on $\mathcal{H}_\omega$ where $\pi_\omega(\mathfrak{A})$ becomes a sub-algebra of $B(\mathcal{H}_\omega)$.
In chapter [5] we made use of the weak topology, we now define the distance function giving the weak topology. We cannot define the weak topology directly on the C^*-algebra because the original definition of the weak topology on bounded operators on a Hilbert space makes use of the scalar product of the Hilbert space. Given a state on the C^*-algebra we can construct a Hilbert space from the algebra, and given this Hilbert space we can use the definition of the weak-topology for $B(\mathcal{H})$

Definition B.5 (Weak topology on $B(\mathcal{H})$) *A neighbourhood of $A \in B(\mathcal{H})$ is in the weak topology on $B(\mathcal{H})$ given by*

$$U_{x_i,y_i,i=1,\ldots,n;\epsilon}(A) = \{B \in B(\mathcal{H}) : |(x_i, (A-B)y_i)| < \epsilon, i = 1, \ldots, n;\ x_i, y_i \in \mathcal{H}\} \tag{B.12}$$

Klein-transformations

The Klein Transformations we used in chapter [5] enable us to change from a set of fields satisfying the abnormal commutation relations to set of fields satisfying normal commutation relations. Details about Klein transformations can be found in [SW64] and in [Jos65].
A subset α of the fields in a (3+1) dimensional field theory satisfies the even-odd rule if every vacuum expectation value with an odd number of fields from α vanishes. The Hilbert space generated from the vacuum by polynomials of the fields in α consists of two orthogonal subspaces $\mathcal{H}_o$ and $\mathcal{H}_e$. Now an operator $p(\alpha)$ can be defined which is 1 on $\mathcal{H}_e$ and -1 on $\mathcal{H}_o$ and using linearity it can be defined elsewhere. The orthogonal subspaces, $\mathcal{H}_o$ and $\mathcal{H}_e$ are invariant under $P_+^\uparrow$ and the operators $p(\alpha)$ commute with the representation of $P_+^\uparrow$. If we have another subset of fields β within the same theory we can define a Klein transformation by

$$\phi'_j = p(\alpha)\phi_j,\ \phi_j \in \beta \tag{B.13}$$

$$\phi'_j = \phi_j,\ \phi_j \notin \beta \tag{B.14}$$

and the following theorem holds

Theorem B.1 (Normal Statistics) *In any* $(3+1)$ *dimensional field theory with abnormal commutation relations there always exists an irreducible set of fields with normal commutation relations obtained from the original set by a Klein transformation.*

Where normal commutation relations hold if tensor fields commute with each other and with spinor fields at space-like separations, and spinor fields anti-commute at space-like separations. And where abnormal commutation relations hold if for subsets of fields one obtains commutation relations and anti-commutation relations where one would expect respectively anti-commutation relations and commutation relations in the case of normal commutation relations.

Bibliography

[AG54] N.I. Achieser and I.M. Glasmann. *theorie der Linearen Operatoren im Hilbert-Raum*. Akademie Verlag (Berlin), 1954.

[ASW84] D. Arovas, J.R. Schrieffer, and Frank Wilczek. Fractional statistics and the quantum hall effect. *Phys. Rev. Lett.*, 53(7):722–723, August 1984.

[AW64] H. Araki and W. Wyss. Representations of canonical anticommutation relations. *Helv. Phys. Acta*, 37:136–159, August 1964.

[BF82] D. Buchholz and K. Fredenhagen. Locality and the structure of particle states. *Commun. Math. Phys.*, 84:1–53, 1982.

[Bor62] H.J. Borchers. On structure of the algebra of field operators. *Il Nuovo Cimento*, XXIV(2):214–236, April 1962.

[Cha87] A.M. Chang. *Experimental Aspects*, chapter 6. In *Graduate Texts in Contemporary Physics* [PG87], 1987.

[DHR69a] S. Doplicher, R. Haag, and J.E. Roberts. Fields obsevables and gauge transformations i. *Commun. Math. Phys.*, 13:1–23, 1969.

[DHR69b] S. Doplicher, R. Haag, and J.E. Roberts. Fields obsevables and gauge transformations ii. *Commun. Math. Phys.*, 15(173-200), 1969.

[FGM] J. Fröhlich, F. Gabbiani, and P.A. Marchetti. Superselection structure and statistics in three-dimensional local quantum theory. *ETH-Zürich-preprint*, (ETH-TH/89-22).

[FK91] J. Fröhlich and T. Kerler. Universality in qauntum hall systems. *Nuclear Physics*, B354:369–417, 1991.

[FM89] J. Fröhlich and P.A. Marchetti. Quantum field theories of vortices and anyons. *Commun. Math. Phys.*, 121:177–223, 1989.

[FS92] J. Fröhlich and U.M. Studer. u(1)×su(2)-gauge invariance of nonrelativistic quantum mechanics, and generalized hall effects. *Commun. Math. Phys.*, 148:553–600, 1992.

[FZ91] J. Fröhlich and A. Zee. Large scale physics of the quantum hall fluid. *Nuclear Physics*, B364:517–540, 1991.

[Hal83] F.D.M. Haldane. Fractional quantization of the hall effect: A hierarchy of incompressible quantum fluid states. *Phys. Rev. Lett.*, 51(7):605–608, August 1983.

[Hal84] B.I. Halperin. Statistics of quasiparticles and the hierarchy of fractional quantized hall states. *Phys. Rev. Lett.*, 52(18):1583–1586, April 1984.

[Has79] P. Hasenfratz. A puzzling combination: Disorder × order. *Physics Letters*, 85B(4):338–346, August 1979.

[JM73] W. Jones and N.M. March. *Theoretical Solid State Physics*, volume 2: Non-equilibrium and Disorder of *Monographs and Texts in Physics and Astronomy*. John Wiley and Sons Ltd., 1973.

[Jos65] R. Jost. *The General Theory of Quantized Fields*, volume 4 of *Lectures in Applied Mathematics*. American Mathematical Society, 1965.

[Koh85] M. Kohmoto. Topological invariant and the quantization of the hall conductance. *Annals of Physics*, 160(2):61–354, April 1985.

[Kuc94] Berndt Kuckert. A new approach to spin and statistics. *Preprint DESY 94-211*, page 15, November 1994.

[Lau81] R.B. Laughlin. Quantized hall conductivity in two dimensions. *Phys. Rev.*, 23(10):5632–5633, 1981.

[Lau83] R.B. Laughlin. Anomalous quantum hall effec: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.*, 50(18):1395–1398, 1983.

[Man68] A.C. Manoharan. Some considerations on configuration space analyticity in general quantum field theory. In W.E. Brittin K.T. Mahantappa, editor, *Mathematical Methods in Theoretical Physics*, volume XI-D of *Lectures in Theoretical Physics*, New York, 1968. Gordon & Breach, Science Publishers, Inc.

[Nak90] M. Nakahara. *Geometry, Topology and Physics*. Graduate Student Series in Physics. Adam Hilger, Bristol and New York, 1990.

[NS83] C. Nash and S. Sen. *Topology and Geometry for Physicist*. Academic Press inc. (London) ltd., 1983.

[NTW85] Q. Niu, D.J. Thouless, and Y-S Wu. Quantized hall-effect as a topological invariant. *Phys. Rev.*, B31:3372–3377, 1985.

[PD57] R.T. Powers and S. Doplicher. On the simplicity of the even car-algebra and free field models. *Commun. Math. Phys.*, 7(1):77–79, 1957.

[PG87] R.E. Prange and S.M. Girvin. *The Quantum Hall Effect*. Graduate Texts in Contemporary Physics. Springer-Verlag New-York, 1987.

[Sch68] L.I. Schiff. *Quantum Mechanics*. International Series in Pure and Applied Physics. McGraw-Hill Book Company, 3 edition, 1968.

[Sto92] M. Stone. *Quantum Hall Effect*. World Scientific, Singapore, 1992.

[SW64] R.F. Streater and A.S. Wightman. *PCT, Spin and Statistics and all That*. The Mathematical Physics Monograph Series. W.A. Benjamin, Inc, New York, 1964.

[SW89] A. Shapere and F. Wilczek. *Geometric Phases in Physics*. Advanced Series in Mathematical Physics, Volume 5. World Scientific, Singapore, 1989.

[TSG82] D. Tsui, H. Störmer, and A. Gossard. *Phys. Rev. Lett.*, 48:1559, 1982.

[vKDP80] K. von Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine structure constant based on quantized hall-resistance. *Phys. Rev. Lett.*, 45:494, August 1980.