Quasi-Particles
for
Fractional Quantum Hall Systems

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Contents

Introduction

Quasi-particles play an important role in physics and form the key to the understanding of many phenomena. The most famous example is found in Landau's Fermi liquid theory, which uses quasi-particles that are 'dressed' versions of the original particles in the system. This theory has been very successful in describing the properties of for example liquid He^3 and normal metals. Another successful theory in which quasi-particles play an important role is BCS-theory of superconductivity in which the excitations over the ground-state are broken BCS-pairs or Bogoliubov quasi-particles. Both theories show that identification of the correct ground-state and the correct quasi-particle excitations simplifies the description of the physics, and allows an description of the phenomenon with only a few effective parameters.

In fractional quantum Hall systems (fqH-systems) many important aspects of the low temperature and low energy dynamics of the system are captured by a one-dimensional edge theory. In lower dimension new possibilities for the nature of quasi-particles open up. Most notably, low dimensional systems can possess quasi-particles with fractional charge and statistics .

The study of these quasi-particles forms an active field of research. However, an approach which describes interactions between two fqH-systems using quasi-particles with fractional quantum numbers has been missing in literature. Until now such interactions were mostly studied within bosonized versions of the theories. In this thesis we will use the quasi-particles with fractional charge and statistics to describe tunneling between two fqH-systems.

The material in this thesis has been organized as follows. To motivate our choice of quasi-particles for fqH-edges, we present some of the general theory of fqH-systems. We use Laughlin's gauge argument to extract the charge structure of the edge and we use Wen's hydrodynamical picture to find the chiral Luttinger liquid which describes fractional quantum Hall edges at simple filling fractions. We give a short introduction to quasi-particle tunneling experiments and theories. For our purposes it is important that these experiments confirm the statements about the charge of the quasi-particles on the fractional quantum Hall edge. We will conclude that it is desirable to see if a theoretical picture which is closer to the usual interpretation of these experiments can be constructed. With the existing bosonic description it is hard to decide whether the correlations in a tunneling current between fqH-edges can be accounted for by unusual properties of quasi-particles.

The first step in the direction of a full quasi-particle description of a fqH-edge is a quasi-particle formulation of the thermodynamics of the $U(1)$ conformal field theory (CFT) which describes an fqH-edge in isolation. We separated this analysis in two parts: first, ch. 2, a state counting procedure and secondly,

ch. 3, a recursive construction of the partition sum. We will give an explicit derivation of the distribution functions of edge quasi-particles using the recursive construction of the partition sum. In principle the distribution functions can be obtained from the state counting procedure, but the recursive construction of the partition sum seems to be the more elegant and the more general method to obtain the distribution functions. The chapter on state counting makes an explicit connection with literature on generalized exclusion statistics and makes clear the conceptual difference between generalized exclusion statistics and the spectral shift statistics found in $U(1)$-CFT.
Then, the second step is to study form factors which describe addition and removal of edge quasi-particles. We used Jack polynomial technology and explicit calculations with the operator algebra to find information about these form factors.
The third and last step is to combine the form factors with the distribution functions to obtain the Green's functions needed to describe finite temperature transport between two edges. This last step we performed numerically for a few examples and for these examples comparison with known exact results shows that the quasi-particle approach is very effective. It gives good approximations to the exact result taking into account only one and two quasi-particle processes.

Our results show that, in fractional quantum Hall systems, quasi-particles with fractional exclusion statistics can be used to do (perturbative) calculations of transport properties at finite temperature and bias voltages. For many similar quantum impurity problems exact solutions are not available. We expect that our approach can be extended to these problems.
One of the most important lessons from the whole analysis is that it is incorrect to think of the fqH-edge quasi-particles as satisfying Haldane's generalized exclusion statistics, which in this context is a too naive generalization of concepts which are applicable to bosons and fermions. Furthermore, it is also misleading to think of these quasi-particles as an ideal gas satisfying a straightforward second quantized description. Instead it is possible to describe the equilibrium behavior of the collective quantum numbers in energy space using distribution functions to indicate the occupation of a collective quantum number.
Another important lesson learned is about the nature of the quasi-particle statistics, for which we propose the terminology spectral shift statistics. It is a new concept of statistics, which has often been called generalized exclusion statistics because it leads to the same state counting. The reason to make the distinction between generalized exclusion statistics and spectral shift statistics is that the conceptual differences between the two are reflected in the physics. The need for a form factor approach is immediately clear from spectral shift statistics whereas generalized exclusion statistics is in essence a second quantized description which shows no direct need for incorporating non-trivial form factors.
Yet another important lesson from this analysis concerns the way the non-equilibrium behavior of two weakly coupled fractional quantum Hall systems can be described within the tunneling Hamiltonian approach. The Green's functions in the two subsystems can be constructed from products of the equilibrium

distribution functions for the collective quantum numbers with irreducible matrix elements $\langle\{\epsilon_i\}|\mathcal{O}|\{\epsilon_i\}\rangle$ of the operator $\mathcal{O} = \Psi^\dagger(\epsilon)\Psi(\epsilon)$. For non-interacting fermions the Green's function would simply reduce to a Fermi-Dirac distribution, but for non-interacting quasi-particles satisfying fractional spectral shift statistics the correct expression will be

$$\langle\mathcal{O}\rangle_{_T} = \sum_{\{\epsilon_i\}}\langle\{\epsilon_i\}|\mathcal{O}|\{\epsilon_i\}\rangle \prod_i n_{_T}(\epsilon_i), \tag{0.1}$$

where $n_{_T}(\epsilon_i)$ is the distribution function for quasi-particle species i. Although this expression looks conceptually simple, the evaluation is difficult because the calculations needed to obtain the (irreducible) matrix elements are involved and we have not yet obtained a complete picture of their mathematical properties. The validity of the quasi-particle approach has been tested in the third step. We calculated approximate electron Green's functions for the $\nu = \frac{1}{3}$ fqH-edge, and we did a more complete calculation for a $\nu' = \frac{1}{2}$ theory, which obeys a much simpler algebra. Although the filling fraction $\nu' = \frac{1}{2}$ is not an allowed fqH-filling fraction it provides a useful testing ground for the quasi-particle approach. As we mentioned before, what we found is that taking into account only one and two quasi-particle processes we already obtained very good agreement of the quasi-particle approximation with the exact result.

1

Fractional Quantum Hall Edges

In this chapter, we provide an overview of some of the existing knowledge on fractional quantum Hall edges and we explain why we developed the formalism that we present in the rest of this thesis.
This chapter starts with a short review of the fractional quantum Hall effect and the theory for the bulk. After this we focus on the edge theories. We determine the charge structure of fractional quantum Hall edges using a combination of well-known gauge arguments [53, 34]. We present Wen's hydro-dynamical picture [81], which leads to the chiral Luttinger liquid description, and we introduce the operators representing the allowed fractionally and integer charged excitations. Then we devote a section to the theoretical and experimental results obtained in the study of point-contacts. We introduce the tunneling Hamiltonian and the expression for the tunneling current we will use in later chapters. We briefly discuss the results obtained by renormalization group studies [46] and exact TBA solutions [22]. In parallel we present the results of tunneling I-V [19, 28] and shot-noise [62, 69] measurements. We conclude with a section in which we explain how the theoretical and experimental results reviewed in this chapter have motivated us to develop the quasi-particle formulation presented in the following chapters of this thesis.

1.1 The Fractional Quantum Hall Effect

In 1982 the fractional quantum Hall effect was discovered by Tsui, Störmer and Gossard [80]. In very clean GaAs-heterostructures they observed a plateau in the Hall-resistance R_H corresponding to a fractional value $\sigma_H = R_H^{-1} = \frac{1}{3}\frac{e^2}{h}$ of the conductance. The next year Laughlin [52] proposed a variational wave-function from which many features of the fractional quantum Hall plateaus at simple filling fractions $\nu = \frac{1}{p}$ could be extracted. The Laughlin wave-function is characterized by its topological order. It can be found using very basic considerations about electrons in the lowest Landau level. At electron densities typical for the simple filling fractions the electrons are restricted to the lowest Landau level because they are separated from the higher Landau levels by the Landau level spacing $\hbar\frac{eB}{m}$, which is of the order of meV for typical magnetic fields of $(1 - 10$ T). This is larger then the energy $k_B T \approx 0.5\mu\text{eV} \ldots 0.1\text{meV}$

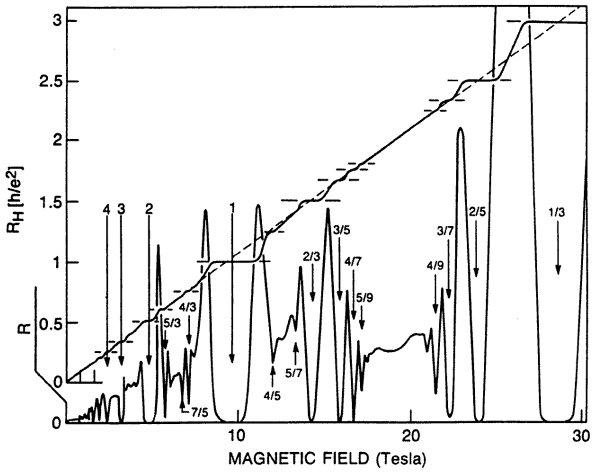

Figure 1.1: Fractional quantum Hall effect, the data show the characteristic plateaus structure in the Hall-resistance R_H and the simultaneous vanishing of the longitudinal resistance R as a function of the magnetic field. Picture taken from [1]

for T in the range 5mK...1K, which are typical temperatures at which these experiments are performed. It is therefore reasonable to assume that the many-body wave-function can be expressed in single particle wave-functions restricted to the lowest Landau level. Furthermore the wave function should be totally antisymmetric and it should be an eigenfunction of total angular momentum. These constraints plus the assumption that the wave function is of the Jastrow-form lead to a variational ground state wave-function [51, 52], the Laughlin wave-function,

$$\Psi_{\frac{1}{p}}(z_1, \ldots, z_N) = \prod_{j<k}^{N} (z_j - z_k)^p \; e^{(-\frac{1}{4}\sum_l^N z_l z_l^*)} \tag{1.1}$$

for the $\nu = \frac{1}{p}$ fractional quantum Hall state.
With the improvement of sample growing technology more and more fractions with odd denominator appeared in the experiments. These quantum Hall states are best described by the Jain hierarchy based on the composite fermion picture [44]. For the composite fermion picture it is assumed that the strongly interacting electron system maps into a system of weakly interacting compos-

ite fermions. A composite fermion is an electron with an even number of flux quanta attached to it. The easiest way to think of this is that the coulomb interaction tries to keep the electrons apart. This creates an area around every electron which is avoided by other electrons. A part of the flux in this area and the electron can now be seen as one particle. If we apply this argument to every electron we get a new system of composites which experience a reduced magnetic field.
The flux attached to an electron is chosen such that the resulting composite is again a fermion. Because of this, these particles will fill composite fermion Landau levels . In general, an integer quantum Hall state for these composite fermions will correspond to a fractional quantum Hall state for the original electrons. If we attach $2m$ flux quanta to the electrons the filling fractions will be related by

$$\nu = \frac{\nu^*}{2m\nu^* + 1} \tag{1.2}$$

where ν denotes the true filling fraction and ν^* denotes the integer filling fraction for composite fermions.
Recently, more exotic plateaus were found at higher filling fractions $\nu > 2$. These plateaus cannot be explained by the composite fermion picture alone. The plateaus at even denominator filling fractions can only be explained by introducing new physical processes like electron or composite fermion pairing. We will not discuss these more exotic quantum Hall states and when we refer to fractional quantum Hall states it will be implicit that we refer to filling fractions $\frac{1}{9} \leq \nu < 1$. The lower bound is introduced here because the two dimensional electron gas will form a Wigner crystal at lower densities.
Laughlin's variational wave function and the composite fermion picture lead to a good description of the physics at the plateaus but they do not explain the fractional quantum Hall effect. To explain it one has to study the effects of disorder in the presence of long range coulomb interactions. In a pure system without disorder the quantum Hall effect would not be observed and the wave-functions proposed above would be good variational ground state only for a discrete set of filling fractions.
That the presence of disorder is crucial can be learned from a closer look at non-interacting composite fermions. If there was only a Landau level structure the Fermi-energy would make discrete jumps if one varies the magnetic field. On lowering the magnetic field some of the composite fermions would move to a higher composite fermion Landau level and give an extra contribution to the conduction. No plateaus would be observed; in fact the conductance would simply be inversely proportional to the magnetic field. However due to the presence of disorder these composite fermions go to localized states at energies neighboring the extended states. If we lower the magnetic field in the presence of disorder the Fermi energy can increase gradually and the composite fermions leaving the extended states are localized and do not contribute to the conduction. The current will be carried by the extended states below the Fermi level, until the moment the magnetic field has changed so much that the Fermi energy reaches the next Landau level.

To explain that in this scenario the conductance will maintain its quantized value over the plateaus the Laughlin gauge argument is needed. This argument shows that the presence of weak disorder does not change the Hall conductance. We will present this argument in the next section, where we discuss the charge structure of the edge.
Following Laughlin's discovery of the variational wave functions, several generalizations [35, 30] preceding Jain's composite fermion picture were proposed to describe plateaus at non-simple filling fractions $\nu \neq \frac{1}{p}$. For these hierarchy schemes Landau-Ginzburg-Chern-Simons theories were proposed by several authors [10, 86, 87, 67, 88, 26, 25]. Blok and Wen [10] showed that for one filling fraction the different hierarchy schemes often lead to the same physics. Furthermore, they used gauge invariance of the fqH-system with boundary to obtain the edge theory directly from the Chern-Simons theory describing the bulk.
All the theories we discussed so far are more or less phenomenological attempts to describe the fractional quantum Hall plateaus. Recently, a unifying field theory for the fractional quantum Hall effect [65, 75, 74] has been developed from microscopic considerations. It takes into account all important aspects: disorder, long-range interactions and strong magnetic field. Some of the most recent experimental surprises in quantum Hall systems, like the continuously varying tunneling exponent, can be understood starting from the insights obtained in this theory.

1.2 Gauge Arguments and Fractional Charge

Gauge arguments play an important role in the theory of the quantum Hall effects. Laughlin used a gauge argument to explain the quantization of the integer quantum Hall effect, and in the first paper mentioning the fractional quantum Hall effect Tsui, Störmer and Gossard used this argument to postulate the existence of fractionally charged excitations. Later Laughlin used a gauge argument to describe how the Laughlin wave function responds to the addition of a flux quantum far away from the position were the flux is added. This gauge argument plus the use of Schrieffer's counting argument led him to propose quasi-particle excitations in the bulk. We will refer to these bulk quasi-particles as Laughlin quasi-particles. In this section we will use the gauge argument to obtain the charge structure of the fractional quantum Hall edge (fqH-edge). The charged edge excitations or edge quasi-particles will be called edge-electron or edge-hole if they have unit charge and quasi-hole or quasi-electron (without the adjective Laughlin) if they have fractional charge. We will use the terminology quasi-particles to refer to all collective particle-like excitations or to refer to the fractionally charged excitations only.
A way to think about the quantization of the conductance [34, 64] and quasi-hole charge is in terms of a so-called ideal system in which a fractional charge is transferred from one edge to the other. This ideal system consists of a two dimensional electron gas (2 DEG) living on a disk. This disk has three rings: an inner ring forming a 2 DEG without disorder, a ring forming a 2 DEG but in

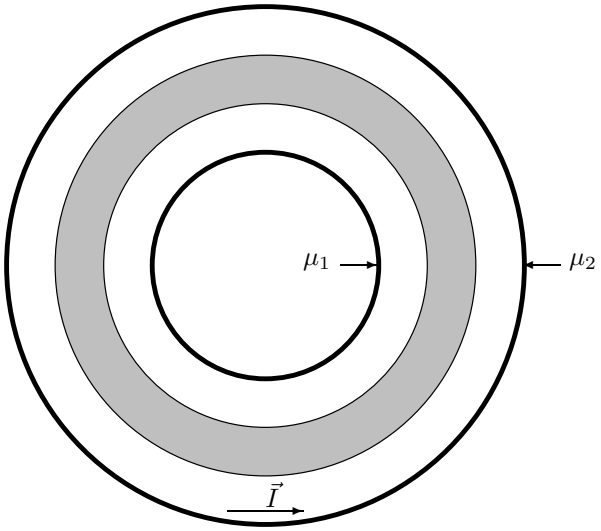

Figure 1.2: Ideal quantum Hall system, a region with disorder protected by two pure guard rings. The μ_i denote electro-chemical potentials and $\vec{I}$ denotes the direction of the current. Picture after [34, 64].

the presence of disorder, and an outer ring forming a 2 DEG without disorder again. Apart from the disorder these rings are the same; they have the same electron density and the magnetic field is constant over the whole sample.

If by tuning the magnetic field we bring the inner and outer rings in the fractional quantum Hall state at filling fraction ν there are two possible results. Either the disorder in the middle ring is large and there are no extended states in this ring or the disorder is milder and there still exist extended states in the middle ring. It is the latter situation in which the middle ring is in a fractional quantum Hall state. For the integer quantum Hall effect Halperin argued that in this situation the system will show the characteristic quantized Hall conductance. He used a gauge argument originally formulated by Laughlin in a geometry without the extra inner and outer guard rings. This gauge argument can be generalized to fractional quantum Hall systems [64] assuming the ground state in the bulk is unique and exhibits a gap. Here we present the argument in a form which closely resembles Laughlin's construction of quasi-particles in the bulk. It will lead us to the charge structure of the edges.

If we adiabatically insert a flux quantum through the inner area, the ground state wave-function will be a ground state of the Hamiltonian during the the process. After the addition the Hamiltonian returns to its original value up to a gauge transformation . However, in the process charge may be transferred from the outer edge to the inner edge or vice versa depending on the sign of the inserted flux. To see how much charge is transfered we first consider how the one electron wave-functions in the lowest Landau level respond to the addition

of flux. If we add a flux ϕ in the hole the single electron eigenfunctions of the angular momentum operator respond as

$$z^m e^{-\frac{1}{4}|z|^2} \rightarrow z^{m+\frac{\phi}{\phi_0}} e^{-\frac{1}{4}|z|^2}, \tag{1.3}$$

and after the addition of a complete flux quantum ϕ_0 we have,

$$z^m e^{-\frac{1}{4}|z|^2} \rightarrow z^{m\pm 1} e^{-\frac{1}{4}|z|^2}. \tag{1.4}$$

We see that the one-particle states have all moved one position in the spectrum of the angular momentum operator. In this situation we can apply Schrieffer's counting argument: the charge transported from one edge to the other will be equal to the average charge per one electron state. Since on average, these states are occupied by ν electrons, we conclude that a charge

$$e^* = \pm \nu e \tag{1.5}$$

is transferred from one edge to the other.
Because we have returned to the old Hamiltonian we have created a new eigenstate of the Hamiltonian. The energy of this state is set by the difference in electro-chemical potential between the inner and outer edge. In principle a calculation of this energy change would involve the Coulomb interactions within the ideal quantum Hall system. However, the simple argument given here gives the correct answer because the gauge argument gives an exact relation between the inserted flux and the charge transfer. We now created a paradox: we can seemingly raise or lower the total energy by transporting electrons from one edge to the other although we assumed the system is in the ground state. The paradox can be resolved by noting that though the bulk is in the ground state the edges are not and it is the uniqueness of the bulk ground state we used to determine how much charge was transported from one edge to the other.
We can now calculate the Hall conductance by relating two expressions for the energy change due to the addition of one flux quantum,

$$\Delta E = -(\mu_2 - \mu_1)\nu e$$

$$\Delta E = -\frac{I}{c}\phi_0 \tag{1.6}$$

where we inserted the flux in the same direction as the existing magnetic field and charge $-\nu e$ is moved from the outer edge to the inner edge.
Identifying $(\mu_2 - \mu_1) = V_H$ we get for the Hall conductance

$$\sigma_H = \frac{I}{V_H} = \nu \frac{ec}{\phi_0} = \nu \frac{e^2}{h} \tag{1.7}$$

and we see that in the presence of the disorder the conductance is still quantized. After insertion of a flux we reobtain the original Hamiltonian for the 2 DEG, and thus we know that the state we obtained is an eigenstate of the Hamiltonian

corresponding to an (de)excitation of the original system before the flux addition. From this argument we see that there are fractional charged excitations at the quantum Hall edges. An energy $\epsilon_1 = -\mu_1 \nu e$ excitation with charge $-\nu e$ at the inner edge and an energy $\epsilon_2 = \mu_2 \nu e$ excitation with charge νe at the outer edge.

1.3 Effective Edge Theories

The low energy dynamics of a quantum Hall system is confined to the edge which exhibits gap-less edge modes. Halperin [35] showed how the dynamics at an integral quantum Hall plateaus can be described by a one dimensional Fermi-liquid theory. Later Landauer and Büttiker [57] gave a formalism to describe measurements on multi-terminal geometries taking the edge channels with quantized conductances as a starting point.
The real breakthrough came in 1990 with Wen's [81] proposal to take the chiral Luttinger liquid (χLL) as an effective theory for the fractional quantum Hall edge. The χLL description predicts electron tunneling exponents, which makes it possible to test its validity in tunneling experiments. We will discuss these tunneling experiments in some detail in the next section.
The predictions of the χLL have been partially confirmed in experiments, but also some serious problems have been found. For the simple filling fractions $\frac{1}{p}$ the χLL is well understood as an effective theory for the edge at the center of a plateaus. For states higher in the hierarchy non-universal conductances are predicted [48] but they are not found in the experiments. The experiments show no qualitative differences between the simple and non-simple filling fractions. Several possibilities to cure this problem were proposed. A cure at a fundamental level was found by Škorić and Pruisken [75] who showed that when the Coulomb forces are taken into account universal behavior of the edges at long distances is obtained.
Furthermore, in experiment it was found that the tunneling exponent shows no plateaus but instead varies linearly with the filling fraction. This behavior was explained in the neighborhood of the center of a integer quantum Hall plateaus [65] and soon extended to fractional quantum Hall plateaus [75]. The structure obtained from the effective edge theory obtained from a Ginzburg-Landau-Chern-Simons theory describing the bulk fractional quantum Hall plateaus will correctly describe the short distance physics of these systems. However in tunneling experiments the long distance physics is probed and the details of the bulk quantum Hall system become important. An analysis of a bulk quantum Hall system with long range disorder shows that the excess charge will be collected in regions with another filling fraction. The coulomb interaction of the edges of these regions with the outer edge give rise to an effective edge theory which is a chiral Luttinger liquid as before but parameterized by the filling fraction instead of the Hall conductance [75].
We will here introduce the χLL along the lines of Wen's [81] hydro-dynamical approach. We pointed out above that there are some limitations to this ap-

proach. For simple filling fractions at the center of a plateaus, however, the validity is undisputed. These simple filling fractions are the main focus of later chapters and a more elaborate discussion of the non-simple filling fractions is not needed here.
At a center of a plateau the bulk of a quantum Hall system is incompressible and low energy excitations are only found at the edge. For the electrons at the edge the drift velocity is given by

$$v = \frac{E}{B}, \tag{1.8}$$

where E is electric field due to a potential confining the electrons to the quantum Hall system. The edge excitations, which are deformations of the edge in the ground state, move with the drift velocity along the edge. If we call the direction along the edge x the deformation of the edge in the perpendicular y direction is described by $h(x)$ which is directly related to the one dimensional excess charge density $\rho(x)$ on the edge $\rho(x) = nh(x)$ where $n = \nu/2\pi l_0^2$ is the two dimensional charge density.
Because of the bulk incompressibility the charge on the edge is conserved and using $j(x) = v\rho(x)$ we get the wave equation

$$\partial_t \rho - v\partial_x \rho = 0, \tag{1.9}$$

which describes deformations of the edge traveling in one direction only. This equation is the chirality constraint. The deformation energy is the work done by moving the excess charge against the electric field E. The effective Hamiltonian for the edge is

$$H = \int dx \tfrac{1}{2} eh\rho E = \int dx \pi \frac{v}{\nu}\rho^2. \tag{1.10}$$

The Fourier modes ρ_k of $\rho(x)$,

$$\rho_k = \frac{1}{\sqrt{2\pi}} \int dx e^{2\pi i \frac{kx}{L}} \rho(x), \tag{1.11}$$

are used to rewrite the conservation law and the Hamiltonian

$$\dot{\rho}_k = \frac{2\pi i}{L} v k \rho_k$$

$$H = \frac{v}{\nu} \sum_{k>0} \rho_k \rho_{-k} \frac{2\pi}{L}. \tag{1.12}$$

If we take ρ_k with $k > 0$ as a canonical coordinate we find the momenta $p_k = \frac{i}{\nu k}\rho_{-k}$. If we apply canonical quantization, i.e. we put $[p_k, \rho_{k'}] = i\delta_{kk'}$, we get the $U(1)$ Kac-Moody algebra describing the edge excitations.

$$[\rho_k, \rho_{k'}] = \nu k \delta_{k+k'} \quad \text{with} \quad k, k' \in \mathbb{Z}$$

$$[H, \rho_k] = vk\tfrac{2\pi}{L}\rho_k. \tag{1.13}$$

Adding an electron to the edge will change the charge on the edge by one unit charge and therefore the electron operator should satisfy

$$[\rho(x), J^{\pm}(x')] = -\delta(x - x')J^{\pm}(x'). \tag{1.14}$$

This leads to the following vertex-operator representation of the electron on the edge

$$J^{-}(x) = e^{-\frac{i}{\sqrt{\nu}}\varphi}, \tag{1.15}$$

where φ is defined by $\rho = \sqrt{\nu}\partial_x\varphi$.
In the gauge argument explaining the exactness of the conductance quantization we found that the edges support fractionally charged excitations $\phi^{\pm}$ with charge $\pm\nu e$,

$$[\rho(x), \phi^{\pm}(x')] = \pm\nu\delta(x - x')\phi^{\pm}(x'). \tag{1.16}$$

These fractionally charged excitations can also be represented by vertex-operators

$$\phi^{\pm}(x) = e^{\pm i\sqrt{\nu}\varphi}. \tag{1.17}$$

A remark is to be made here. The fractionally charged excitations always come in clusters of integer total charge, and to have a fractional charge on one edge it should be connected by a fractional quantum Hall bulk to another edge with an opposite fractional charge. An example of such a situation is found in the Corbino disc geometry in the gauge argument after transfering one charge from the outer edge to the inner edge.
Using the propagator for the φ,

$$\langle\varphi(x,t)\varphi(0,0)\rangle = -\ln(x - vt) + \text{const.} \tag{1.18}$$

the electron and the quasi-particle propagators at zero temperature can be calculated directly

$$\begin{aligned}\langle J^{+}(x,t)J^{-}(0,0)\rangle &= \frac{1}{(x - vt)^{\frac{1}{\nu}}},\\ \langle\phi^{+}(x,t)\phi^{-}(0,0)\rangle &= \frac{1}{(x - vt)^{\nu}}.\end{aligned} \tag{1.19}$$

From the form of the electron propagator we can see that it only describes an electron satisfying fermionic exchange statistics correctly for simple filling fractions $\frac{1}{p}$ with p odd. For other filling fractions the vertex operator is not fermionic and extra bosonic fields are needed to describe the electron correctly.

1.4 Inter Edge Tunneling

In standard two or four terminal measurements of the quantum Hall effect chiral Luttinger liquid properties are probed only indirectly. These measurements

amount to measuring the Hall conductance and do not probe the power law behavior of the Green's functions typical for Luttinger liquid behavior. It is, however, possible to test this power law behavior in tunneling experiments which probe the dynamics of particles settling into the edge.
Point contacts are often used for tunneling experiments. Because of their simplicity they form an ideal theoretical and experimental testing ground in mesoscopic physics. In the point-contacts we will discuss in this section two fqH-edges are brought close together. They interact via tunneling processes. The χLL physics expresses itself in the non-linear power-law dependence of the tunneling current on the bias voltage, gate voltage and temperature [81, 46, 22]. The charge of the tunneling quasi-particles expresses itself in the current-fluctuations in the low temperature regime (shot-noise) and we will argue that the crossover regime between shot-noise and thermal noise can only be described taking into account the statistics and the correlations of the quasi-particles.

1.4.1 Tunneling Hamiltonian and IV-characteristics

Our starting point for an analysis of tunneling is the tunneling Hamiltonian [4, 7]. The evaluation of the current to lowest order in the transmission amplitudes gives rise to an equation for the tunneling current which shows that for low bias voltages the behavior of a point-contact can be described by the edge Green's functions of the tunneling particles. The tunneling Hamiltonian contains three parts of which two parts H_1 and H_2 describe the two uncoupled subsystems and the third part is the interaction H_t term,

$$H = H_1 + H_2 + H_t. \tag{1.20}$$

The Hamiltonians H_i describe χLL 's at filling ν_i and are copies of the Hamiltonian describing a single edge in isolation, eq. 1.10,

$$H_i = \frac{v_i}{\nu_i} \sum_{k>0} \frac{\rho_k^i \rho_{-k}^i}{L}. \tag{1.21}$$

The tunneling Hamiltonian will depend on whether we tunnel through the bulk of a quantum Hall system, where the tunneling quasi-particles will be the fractionally charged bulk excitations

$$H_{t,qh} = \sum_{kq} \{R_{kq} \phi_k^{1,-} \phi_q^{2,+} + R_{kq}^* \phi_q^{2,-} \phi_k^{1,+}\}. \tag{1.22}$$

or through 'vacuum' i.e. a barrier which is not a quantum Hall system where the tunneling current will exist of electrons,

$$H_{t,e^-} = \sum_{kq} \{T_{kq} J_k^{1,-} J_q^{2,+} + T_{kq}^* J_q^{2,-} J_k^{1,+}\}. \tag{1.23}$$

$H_{t,qh}$ is a good operator only if both sides of the barrier are at the same filling fraction and cannot be used in the more general situation with a mismatch in

the filling fractions when no natural notion of tunneling through the bulk exists. The $J_k^{\pm}, \phi_k^{\pm}$ are the Fourier modes of the vertex operators we encountered before. Now the current can be obtained from expanding

$$I(t) = -e\langle \dot{Q}_1(t) \rangle \tag{1.24}$$

with $\dot{Q}_1 = [H, Q_1]$ and Q_1 the charge operator which in Fermi liquids would be equivalent to $-e\sum_k \Psi_k^- \Psi_{-k}^+$ with $\Psi_k^{-(+)}$ the Fourier modes of the electron creation(annihilation) operator. In Fermi-liquids this operator is equal to the number of particles times the charge, in non-Fermi liquids the same operator cannot be interpreted as a counting operator. This is an issue we will return to in later chapters were will try to express $\langle J_k^- J_{-k}^+ \rangle$ using edge electron and quasi-hole distribution functions. The interpretation of this operator plays a crucial role in interpreting the expression for the DC part of the tunneling current to lowest order in T_{kq} ,

$$I(V,T) \propto -e|t|^2 \int_{-\infty}^{\infty} d\epsilon \, \{g_1(\epsilon - eV)G_2(\epsilon) - G_1(\epsilon - eV)g_2(\epsilon)\} \ , \tag{1.25}$$

where it is assumed that $T_{kq} = t\delta_{kq}$ is independent of k and q. The Green's functions are given by

$$\begin{aligned} g_i(\epsilon) &= \langle J_\epsilon^{i,-} J_\epsilon^{i,+} \rangle_T, \\ G_i(\epsilon) &= \langle J_\epsilon^{i,+} J_\epsilon^{i,-} \rangle_T. \end{aligned} \tag{1.26}$$

For describing quasi-hole tunneling we have to replace the electron Green's functions with the quasi-hole Green's functions $h(\epsilon)$ and $H(\epsilon)$ and replace $-e$ by $e^* = \nu e$ the quasi-particle charge,

$$\begin{aligned} h_i(\epsilon) &= \langle \phi_\epsilon^{i,+} \phi_\epsilon^{i,-} \rangle_T, \\ H_i(\epsilon) &= \langle \phi_\epsilon^{i,-} \phi_\epsilon^{i,+} \rangle_T. \end{aligned} \tag{1.27}$$

For chiral Luttinger liquids the zero temperature Green's functions can be calculated from expression 1.19 by a Fourier transformation,

$$G_i(\epsilon) \propto \epsilon^{\nu^{-1}-1}, \tag{1.28}$$

which illustrates the power-law behavior.
In experiments in a cleaved edge over-growth geometry, in which the electrons tunnel from a Fermi-liquid (a normal metal) into a fqH-edge, the parameter characterizing the chiral Luttinger liquid was measured. Contrary to theoretical expectations it was found that this parameter depends continuously on the magnetic field strength or the true filling fraction. Where a measurement of the Hall conductance shows plateaus, see figure: 1.3, the measurement of the tunneling exponent shows no plateau structure. This measurement clearly shows that the applicability of the hydro-dynamical approach can only be valid at the center of a plateau.

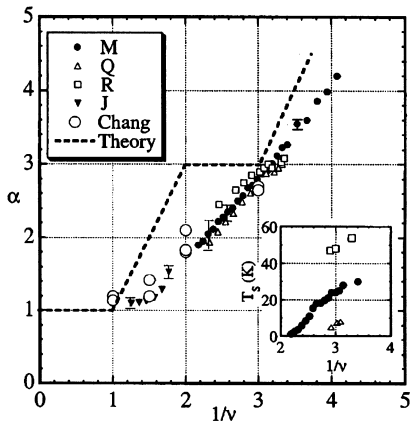

Figure 1.3: Tunneling exponent as function of the magnetic field, picture taken from reference [28]. The measurement was performed in cleaved edge overgrowth geometry, in which the electrons tunnel from a Fermi-liquid into a fqH-edge. The dashed line was taken from [73].

The temperature and bias voltage dependence of these point-contacts can be derived from renormalization group (RG) analysis of Luttinger liquids [46] where one finds that the tunneling rate from a Fermi liquid into a chiral Luttinger liquid drops to zero as temperature is brought to zero. This is due to the orthogonality catastrophe: the electrons in the ground state of the χLL are strongly correlated and adding or removing electrons gives rise to an excited state of the remaining electrons orthogonal to their ground state.

For the RG-analysis of tunneling between two equivalent edges, see figure: 1.4 ($\nu_1 = \nu_2 = \nu$) Kane and Fisher [46] took in to account n electron processes or, in the opposite limit, n quasi-particle processes. This allowed them to decide which of these processes are relevant perturbations of the theory of uncoupled edges. It turns out that for filling fractions $\nu > \frac{1}{4}$ the 1 quasi-particle tunneling processes are the only relevant perturbation. For lower filling fractions also multi quasi-particle processes contribute. This puts a practical limit on the applicability of the results we derived from the tunneling Hamiltonian. When we go to lower filling fractions we need to add extra terms to the tunneling Hamiltonian $H_{t,qp}$ to take into account the extra processes.

The RG-analysis of quasi-hole tunneling, fig. 1.4(a), leads to the following expressions: the tunneling current at zero temperature

$$I_{qp}(V) = \frac{e^2}{h}\left[\sum_{n=1}^{\infty} a_{nV}|v_n|^2 V^{2\rho^2\nu-1}\right], \tag{1.29}$$

and the conductance at finite temperature

$$G_{qp}(T) = \frac{e^2}{h}\left[\sum_{n=1}^{\infty} a_{nT}|v_n|^2 T^{2(\rho^2\nu-1)}\right] \tag{1.30}$$

where a_{nV}, a_{nT} are known constants [46] and the v_n are complex numbers depending on the details of the point-contact.
For electron tunneling, fig. 1.4(b) similar formulas are found : the zero-temperature current-voltage relationship

$$I_g(V) = \sum_{n=1}^{\infty} b_{nV}|t_n|^2 V^{2\frac{\rho^2}{\nu}-1}, \tag{1.31}$$

and the conductance at finite temperature

$$G_e(T) = \frac{e^2}{h}\left[\sum_{n=1}^{\infty} b_{nT}|t_n|^2 T^{2(\frac{\rho^2}{\nu}-1)}\right] \tag{1.32}$$

where b_{nV}, b_{nT} are known constants [46] and the t_n are complex numbers depending on the details of the point-contact.

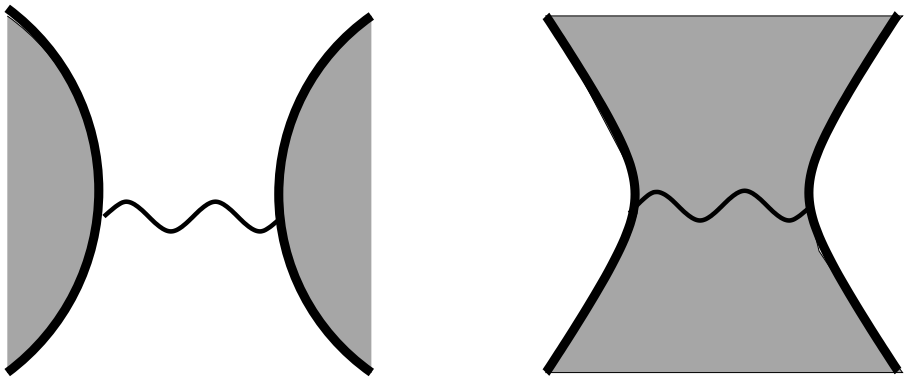

Figure 1.4: Tunneling through vacuum (a) and through the bulk (b), the shaded area denotes quantum Hall bulk.

When temperature is lowered sufficiently this RG-analysis shows that the quasi-hole tunneling interaction between two edges of the same quantum Hall system becomes so strong that perturbation theory breaks down because the $n = 1$ contribution diverges. The interaction will split the system in two parts, see figure: 1.4. This strong coupling or low temperature regime, see figure: 1.4(a),

can be described by electron tunneling through vacuum between between the edges of the two newly formed fqH-systems.
At $\nu = \frac{1}{3}$ only 1 quasi-particle processes are relevant, and therefore, it corresponds to an exactly solvable model [22]. The exact solution obtained from thermodynamic Bethe ansatz describes the crossover between the quasi-particle tunneling and the electron tunneling regimes. Furthermore the exact solution shows a duality relating the low temperature, strong coupling, electron tunneling regime to the high temperature weak coupling quasi particle tunneling regime.

1.4.2 Shot-Noise

Although difficult experimentally it is possible to measure the fluctuations in a quasi-hole tunneling current. The noise power

$$S(\omega) = \int dt e^{i\omega t} \tfrac{1}{2} \langle \{ I(t), I(0) \} \rangle. \tag{1.33}$$

contains two contributions: the thermal or Johnson-Nyquist noise S_{JN} and the partition or shot noise S_{SN}. Pure thermal noise in one edge can easily be calculated from the partition sum describing the chiral Luttinger liquid on the edge. It leads to a confirmation of the fluctuation dissipation theorem,

$$S_{JN}(T) = 2\nu T. \tag{1.34}$$

Also the shot noise in the limit of low temperature and small tunneling currents, where the particles tunnel one by one, can easily be calculated and leads to the famous charge spectroscopy result

$$S_{SN}(V) = 2e^* I_B(V)(1-t) \tag{1.35}$$

where $t << 1$ is the transmission. $S_{JN}(T)$ is the zero frequency noise power at temperature $T >> eV$ and $S_{SN}(V)$ is the zero frequency noise power at $T << eV$. For tunneling through a $\nu = \frac{1}{3}$ fractional quantum Hall bulk a charge $\frac{e}{3}$ for the tunneling quasi-particles is found in the experiment, see figure 1.5. The constant transmission in the shot-noise experiments, which would not be expected from the non-linear I-V-characteristic we described above, is due to tuning of the point-contact. For different values of the bias voltage the gate-voltage was tuned to obtain the same transmission.
To describe the crossover between the thermal noise and shot noise regime is a problem which is still largely unsolved. For the tunneling through a filling fraction $\nu = \frac{1}{3}$ fractional quantum Hall state an exact solution is available [22, 23] but for general filling fractions $\nu < \frac{1}{4}$ the problem is no longer exactly soluble.
For bosons and fermions the crossover from shot-noise to thermal noise takes the form [57],

$$S(T) = 2\frac{e^2}{h} \sum_{i,j=1,2} \int dE \ \mathrm{Tr}(A_{ij} A_{ji}) f_i(E, V_i)(1 \mp f_j(E, V_j)) \tag{1.36}$$

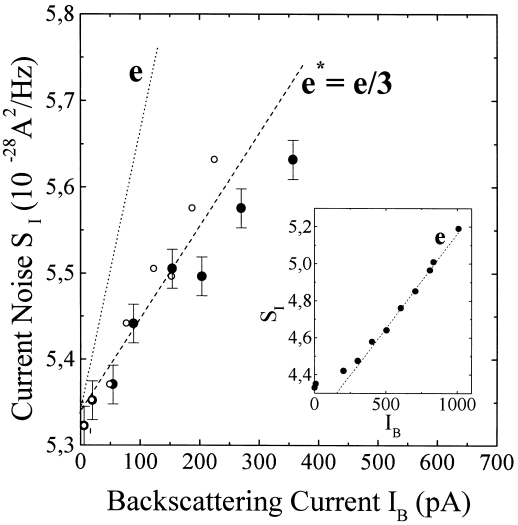

Figure 1.5: The noise power as a function of the tunneling current, figure taken from: [69]. The slope is proportional to the quasi-particle charge. The inset shows the noise power for electron tunneling through vacuum.

where the f_i are distribution functions in the two edges connected by the point contact and the $+$ and $-$ refer to Bose and Fermi statistics, respectively. The A_{ij} are derived from the scattering matrix s_{ij} characterizing the point contact. The behavior under temperature or bias-voltage changes is contained in the factor $f_i(E, V_i)(1 \mp f_j(E, V_j))$ and depends only on the exclusion statistics of the particles. This suggests that if the quasi-holes possess fractional statistics this might be visible in the crossover behavior of the zero frequency noise. An attempt to make the dependence on the fractional statistics explicit [41] does not account for the correlations between the quasi-holes properly, and therefore, the results can not be applied to fractional quantum Hall edges. How to generalize equation 1.36 to quasi-holes is at the moment unclear because the properties of creation and annihilation operators used in the standard second quantized formalism for fermions or bosons can not be used for quasi-holes. Therefore, the usual non-equilibrium techniques like Keldysh or Landauer-Büttiker can not be applied.

1.5 Edge Quasi-Particles

The gauge argument gives us strong evidence that the charge on the edge is quantized in units $e^* = \nu e$. The shot-noise experiments confirm this hypothe-

sis, because they show the current possesses a granularity consistent with the assumption of this charge quantization. They also show that a tunneling current through the bulk can support this granularity, where a current through vacuum can only support the electron charge.
As we pointed out before, most calculations on tunneling processes are done within a bosonic formulation. Although they give the correct results one would like to construct a calculational scheme which is closer to an interpretation in terms of fractionally charged quasi-particles. If it is correct, such a scheme should have certain advantages over others. A good criterion could be whether it is able to describe the crossover from shot noise to thermal noise.
In the chiral Luttinger liquid the charged particles are represented by vertex-operators as we have seen in section 1.3 when we described the edge-electron and quasi-hole operators. The behavior of these operators can be analyzed with techniques from conformal field theory which in general is not formulated in a traditional condensed matter language. We try to get closer to condensed matter formulation like Landauer-Büttiker when we try to describe two edges in contact using quasi-particle occupation numbers. However, a Landauer-Büttiker type of approach depends to much on fermionic statistics and we have to take at least a few steps back and start from a tunneling Hamiltonian description which is not based on fermionic or bosonic statistics and therefore a good starting point for our analysis.
To analyze the tunneling Hamiltonian one usually goes to a formulation in k-space. At low temperatures the only states important for transport through a tunneling barrier will then be those states close to the Fermi surface. In conformal field theory the transition to k-space corresponds to a transition from vertex-operators to their modes. Now, an important observation will be that the quasi-particles develop a Fermi-surface. This will be very helpful for the analysis of tunneling processes because it will strongly reduce the phase space in which many quasi-particle processes are important. The question then is how tunneling quasi-particles, edge electrons or quasi-holes, leave from and settle in an edge. We know already that in (chiral) Luttinger there is the orthogonality catastrophe which leads to strong suppression of tunneling at low temperatures. For the calculation of Green's functions at finite temperatures we will need quantitative information on these tunneling processes which is given in the language of the quasi-particle picture. To justify the quasi-particle picture this point needs to be settled, and using only processes involving a few quasi-particles, it has to lead to a good approximation to the experimental data.

2

Spectral Shift Statistics

In this chapter we present an interpretation of the state counting formula introduced by Haldane and Wu [31, 42] which differs from Haldane's original interpretation. For the description of systems which satisfy Haldane-Wu state counting with only one quasi-particle species present, like some 1D systems solvable by TBA methods and conformal field theories, the new concept of spectral shift statistics seems to be the correct one.

We will first present Haldane-Wu state counting and Haldane's original interpretation. Then we will argue that this interpretation has in it a hidden second quantized description for the quasi-particles, which makes it suited for the study of systems where interactions or geometry restrict the number of quasi-particles occupying a number of states which share most of their quantum numbers. In the second section we will discuss the state counting in fractional quantum Hall systems with Laughlin quasi-particles. Then we present the main topic of this chapter, spectral shift statistics, and we will give a proof of the duality between g and g^{-1} spectral shift statistics. We conclude with a section which discusses the difference between generalized exclusion statistics and spectral shift statistics. In the same section we also argue that both types of statistics have their own arena of applications.

2.1 Haldane-Wu State Counting and Generalized Exclusion Statistics

In Bethe-Ansatz a quantum field theory problem is mapped to a set of quantum mechanics problems describing a specified number of quasi-particles. In the quantum mechanics problems it is possible to count the number of states. Many of the models solved by Bethe-Ansatz showed the same type of relation between number of states and the quasi-particle content of the system [31]. This Haldane-Wu state counting was found in many two and one dimensional systems, such as: anyons in the lowest Landau level [20], Calogero-Sutherland models [38, 49, 20], fractional quantum Hall systems [31, 45], Haldane-Shastry spin chains and some other models solvable by conformal field theory or TBA methods.

The information needed to do the counting is summarized in the statistics matrix

G whose entries g_{ij} appear in the counting formula,

$$\#_{\mathrm{G}}(\{D_i\},\{N_i\}) = \prod_i \frac{(D_i + N_i - 1 - \sum_j g_{ij}(N_j - \delta_{ij}))!}{N_i!(D_i - 1 - \sum_j g_{ij}(N_j - \delta_{ij}))!}, \tag{2.1}$$

where N_i denotes the number of particles of species i, D_i the dimension of the one-particle Hilbert space for species i, and $\#_{\mathrm{G}}(\{D_i\},\{N_i\})$ the number of many-body states.
In the years following the introduction of Haldane-Wu state counting the thermodynamics of this kind of particles were studied, and in 1994 several authors [42] simultaneously obtained the distribution functions n_i which in their most general form are given implicitly by the Isakov-Ouvry-Wu equations,

$$\left(\frac{\lambda_i - 1}{\lambda_i}\right) \prod_j \lambda_j^{g_{ij}} = z_i \;\text{ with }\; z_i = e^{-\beta(\epsilon_i - \mu_i)}$$

$$n_i(\epsilon) = z_i \partial_{z_i} \ln \prod_j \lambda_j \tag{2.2}$$

In the same year a duality relating g and $\tilde{g} = g^{-1}$ statistics was found [61],

$$g\, n_g(\epsilon) = 1 - \tilde{g}\, n_{\tilde{g}}(-\tilde{g}\epsilon). \tag{2.3}$$

Alternative formulations where studied [66, 59] which allow for application of the statistics to small systems [66] important for the application to mesoscopic systems and systems without a high energy cut off [59].
To explain the state counting Haldane [31] introduced the concept of generalized exclusion statistics in which he generalized the Pauli exclusion principle:

Generalized exclusion statistics is the following interpretation of Haldane-Wu state counting: The quantity $D_i - 1 - \sum_j g_{ij}(N_j - \delta_{ij})$ represents the number of states accessible for the N_i-th particle of species i in the presence of N_j particles of the species $j \neq i$.

This interpretation can be rephrased in a more precise language which leads to two properties [37] which are satisfied by systems with generalized exclusion statistics:

1 State Homogeneity: The number of states accessible for the N_i-th particle of species i in the presence of N_j particles of the species $j \neq i$ is independent of the state of the particles already present.

2 Particle Homogeneity: The change of the number of accessible states is independent of the number of particles , i.e. $G_N - G_{N-1} = G_{N'} - G_{N'-1}$ with $N - 1$ and $N' - 1$ allowed particle numbers.

If we rethink these properties there is one important observation to be made. Implicit in the definition of generalized exclusion statistics, is the assumption

that, modulo quantum indistinguishability, all the states created are independent. When we analyze the conformal field theory we will notice that the states created are dependent which means that using the number of states accessible for the N_i-th particle of species i in the presence of N_j particles of the species $j \neq i$ to obtain the state counting would lead to overcounting of the number of states.

2.2 Fractional Statistics for Bulk Quasi-particles

As an example of Haldane-Wu state counting we will discuss here briefly the state counting for bulk quantum Hall states, which was obtained as a result of numerical studies. For completeness we start with a short review of the properties of bulk quasi-particles.
If we insert a flux through an infinitely small solenoid somewhere in the bulk of a quantum Hall system and perform a singular gauge transformation, the same reasoning we performed in chapter 1 to find the quantization of the Hall conductance will show that we created an exact excitation over the bulk ground-state with fractional charge $\pm\nu e$.
These Laughlin quasi-particles carry a fractional charge and satisfy Abelian braid-statistics [8]. Braid-statistics was proposed earlier in another context by Leinaas and Myrheim. It describes the effects of interchanging particles on the many particle wave function. The exchange of two particles leads to a phase factor, but, contrary to fermionic or bosonic statistics, we have to specify the path by which the particles are exchanged. This is because in two dimensions it is no longer possible to deform a given path into another path by continuous deformation. Instead it can be shown that the paths fall into equivalence classes which are in one to one correspondence with elements of the braid group.
In [8] the phase factors associated with the exchange of two Laughlin quasi-holes were related to the filling fraction,

$$\tau_{ij} = e^{\pi\nu}, \quad \tau_{ij}^{-1} = e^{-\pi\nu}, \tag{2.4}$$

where τ_{ij} corresponds to an clockwise exchange of a quasi-hole i with a quasi-hole j and τ_{ij}^{-1} with an anti-clockwise exchange.
For fermions and bosons the exchange properties are directly related to the rules for constructing many particle states and naturally the question arises whether such a connection exists for particles with braid statistics. In general such a connection cannot be made. In some special cases more insight in the properties of these particles has been gained. The most notable example is the case of anyons confined to the lowest Landau level [20, 42], which can be mapped to a Calogero-Sutherland model. In this special case a direct connection was made between the exchange statistics parameter and the state counting.
The construction of many quasi-particle state in the bulk has been studied by several authors [31, 42, 45, 17, 40, 85, 79]. It turns out that the state counting for many-particle states of only quasi-holes or only quasi-particles could be found easily but the state counting for many-particle states with both quasi-holes $(-)$

and quasi-particles (+) present is harder to understand. It was found that the bulk quasi-particles satisfy Haldane-Wu state counting 2.1 and the correct statistics matrix for the $\nu = \frac{1}{2m+1}$ fractional quantum Hall state was found by numerical studies [45, 17, 79] on small systems of six electrons and $\pm 12m + 6$ flux quanta to be,

$$G = \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--} \end{pmatrix} = \begin{pmatrix} 2 - \frac{1}{2m+1} & -2 + \frac{1}{2m+1} \\ 2 - \frac{1}{2m+1} & \frac{1}{2m+1} \end{pmatrix}. \tag{2.5}$$

Where for large N_ϕ the number of fluxes is approximately $D_+ = D_- = \frac{N_\phi}{2m+1}$. More general a similar result for the Jain series with $\nu = \frac{\nu^*}{2\nu^* m+1}$

$$G = \begin{pmatrix} 1 + \frac{2m}{2\nu^* m+1} & -1 - \frac{2m}{2\nu^* m+1} \\ 1 + \frac{2m}{2\nu^* m+1} & \frac{2(\nu^*-1)m+1}{2\nu^* m+1} \end{pmatrix}. \tag{2.6}$$

was found [40] from numerical studies, see figure 2.1.

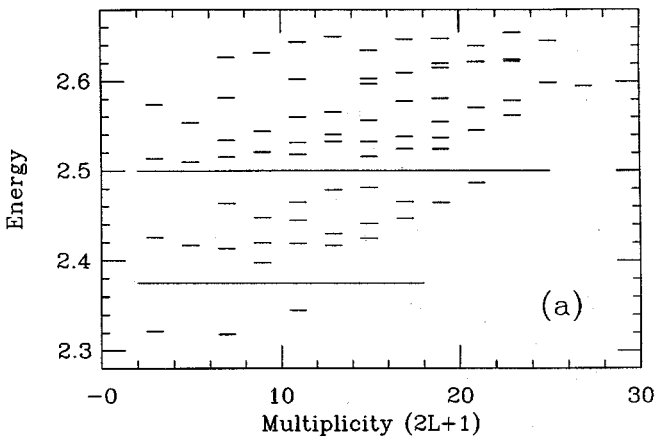

Figure 2.1: **Numerical state counting:** Low-lying energy levels for 5 electrons on a sphere with $N_\phi = 14$-flux quanta which is close to $\nu = \frac{1}{3}$. The solid lines separate the lowest band with two quasi-holes from the first excited band with three quasi-holes and one quasi-particle and the latter band from higher states. For this geometry $D_- = \frac{n_\phi + \nu^* + 4m}{2\nu^* m+1} + \nu^* - 1$ and in this case $D_- = \frac{19}{3}$ where $N_- = 2$ and $N_+ = 0$. Using these numbers the correct number of states is found $21 = 11 + 7 + 3$. Figure taken from [40].

An important point to put forward here is that the state counting is obtained from counting the number of states for a given number of quasi-particles and a given system size. This may seem a minor point but as we will see in later sections when we investigate the nature of statistics in fractional quantum Hall

edges this method to determine the state counting does not reveal the nature of the generalized statistics. Although the state counting is clearly consistent with Haldane-Wu state counting it remains unclear what the nature of this statistics is, because the spectral shift statistics and Haldane's generalized exclusion statistics lead to the same state counting.

2.3 State Counting in Conformal Field Theories

In this section we will argue how to obtain Haldane-Wu state counting from the conformal field theory and propose a general principle which also applies to some TBA systems [63].

As we shall discuss in the next chapter a fractional quantum Hall edge at filling $\nu = \frac{s}{r}$ with $r = p$ and $s = 1$ gives rise to a $U(1)$ conformal field theory with a basis of quasi-particle states of the form

$$\begin{aligned} & J_{-(2M-1)\frac{r}{2s}+Q-m_M} \cdots J_{-\frac{r}{2s}+Q-m_1} \phi_{-(2N-1)\frac{s}{2r}-\frac{Qs}{r}-n_N} \cdots \phi_{-\frac{s}{2r}-\frac{Qs}{r}-n_1} |\, Q \,\rangle \\ & \quad \text{with} \quad m_M \geq \ldots \geq m_1 \geq 0, \quad n_N \geq \ldots \geq n_1 \geq 0 \;, \\ & \qquad n_1 > 0 \quad \text{if} \quad Q < 0 \;. \end{aligned} \tag{2.7}$$

In fact this list of states can generally be used to describe a $U(1)$ conformal field theory at compactification radius $R = \sqrt{\frac{r}{s}}$ with s and r co-prime and $s \leq r$. This list is special because the construction of the J-quanta part of these quasi-particle states is independent of the ϕ-quanta part. If we add ϕ-quanta or J-quanta with other mode indices then given above we will create null states or states in the linear span of the given basis states. Completeness of the basis is ensured by a one-to-one correspondence between the basis states and Jack polynomials [43].

We first reformulate the rules for constructing a quasi-hole state,

$$\begin{aligned} & \phi_{-(2N-1)\frac{s}{2r}-\frac{Qs}{r}-n_N} \cdots \phi_{-\frac{s}{2r}-\frac{Qs}{r}-n_1} |\, Q \,\rangle \\ & \quad \text{with} \quad n_N \geq \ldots \geq n_1 \geq 0 \;, \\ & \qquad n_1 > 0 \quad \text{if} \quad Q < 0 \;, \end{aligned} \tag{2.8}$$

into a form more suitable for a graphical representation. We see that if the n-th quasi-hole added carries a mode index $l_n = -\epsilon_n$ corresponding to a dimensionless energy ϵ_n the next quasi-hole can be added at an energy ϵ_{n+1} from the set $\{\epsilon_n + \frac{s}{r} + k\}$ with k a non-negative integer.

The relation

$$\epsilon_{i+1} \in \left\{ \epsilon_i + \frac{s}{r} + k_i \right\} \quad \text{with} \quad k_i = n_i - n_{i-1}, k_1 = n_1 \in \mathbb{N} \tag{2.9}$$

defines what we call spectral shift statistics. The spectrum for the next particle to be added is shifted by $\frac{s}{r}$ compared to the spectrum of the last particle added and only mode-indices with a higher mode-index then those of the particles already present in the system are allowed. Spectral shift statistics is not a rule which tells you which mode-indices are available for adding particles to the system but it describes how to construct a complete set of linear independent many-particle states. The problem whether a particle can be added to or extracted from a state at a certain energy is a separate problem and cannot be answered on the basis of the statistics alone.

To obtain the state counting we will now introduce a pictorial representation of a N-quasi-hole state. Next to each other we draw the N different spectra, that is we draw all the possible mode indices. We start on the left with mode-indices $-\frac{s}{2r} - \frac{Qs}{r} - n_1$ next to it on the right we draw the mode-indices $-3\frac{s}{2r} - \frac{Qs}{r} - n_2$ and in the n-th column we draw the mode-indices $-(2N-1)\frac{s}{2r} - \frac{Qs}{r} - n_N$. Because the $n+r$-th spectrum falls on top of states in the $n-th$ spectrum we only draw the first $r+1$ spectra, the last spectrum we draw is drawn with open dots to indicate that the points there should be identified with points in the first spectrum with which they share the same mode-index.

In this picture we can draw a state by drawing k_1 up-arrows in the first column connecting neighboring mode-indices and ending at the point denoting the mode-index of the first particle. At this point we draw a right-arrow to the point denoting the lowest mode index allowed for the second particle. We can repeat the procedure for the second particle. We draw k_2 up-arrows and from the point denoting the second mode index we draw a right-arrow to the point denoting the lowest mode index allowed for the third particle. We can continue this way until we represented all N-particles.

If we count all possibilities to place N quasi-holes in a truncated spectrum containing all mode indices corresponding to energies up to $(2N-1)\frac{s}{2r} + \frac{Qs}{r} + l$ we have to take into account that the last quasi-hole is not always placed at the highest possible mode-index inside the truncated spectrum. In the picture we have drawn up-arrows following the last particle to allow for such a situation. The number of N-quasi-hole states in this truncated spectrum is now simply given by the number of possible configurations of arrows representing an N-quasi-hole state. We have to divide out permutations which work on one type of arrows only because the particles (quasi-holes) and holes in the quasi-hole spectrum are indistinguishable and we find

$$\#_{\frac{s}{r}}(N,l) = \frac{(N+l)!}{N!l!}. \tag{2.10}$$

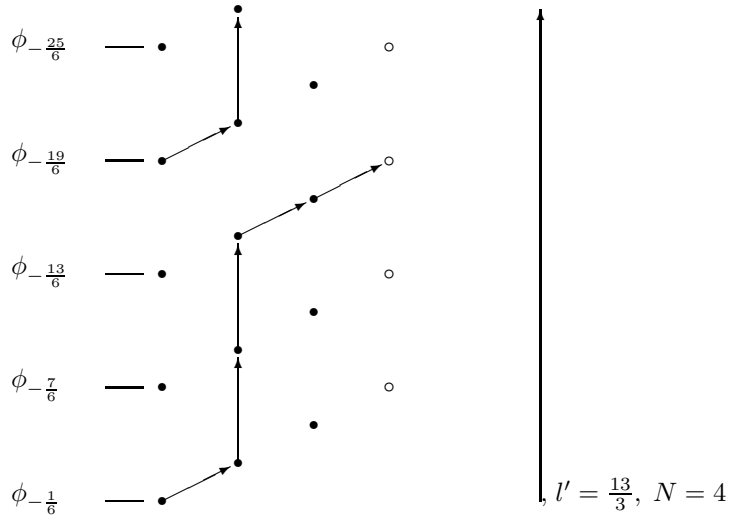

Figure 2.2: Four quasi-hole state, $\phi_{-\frac{19}{6}}\phi_{-\frac{17}{6}}\phi_{-\frac{15}{6}}\phi_{-\frac{1}{6}}|Q=0\rangle$ with $\nu = \frac{1}{3}$

If we move the N-dependent part of our highest mode-index into l we get $l' = l + N\frac{s}{r}$

$$\#_{\frac{s}{r}}(N, l') = \frac{(N + l' - N\frac{s}{r})!}{N!(l' - N\frac{s}{r})!}, \tag{2.11}$$

and we can see that l' is the size of the interval in mode-index space to which we restrict our mode indices.
If $N \bmod r = 1$, which implies that the state to which the last quasi-particle is added carries integer excess charge , we can identify $(N-1)\frac{s}{r} + l + 1$ with the number of one-particle states in this interval of the spectrum. After this identification we recognize Haldane-Wu state counting with $l' = D + (\frac{s}{r} - 1)$ where D denotes a number of one-quasi-hole states. If $N \bmod r \neq 1$ this identification leads to a fractional D which cannot be interpreted as the number of one-quasi-hole states. That is why we will use l' except when we like to compare our results with exclusion statistics.
If we want to keep l' fixed and study the number of many quasi-holes states in the truncated spectrum for another particle number N' then N' should satisfy

$$\exists k \in \mathbb{N} :\ (N'-1)\frac{s}{r} + k = l' \tag{2.12}$$

and we can apply equation 2.11 with N replaced by N'.
Lets turn our attention to the edge electrons. The important difference with the quasi-holes is that the spectral shift parameter $g = \frac{r}{s} \geq 1$.

$$J_{-(2M-1)\frac{r}{2s}+Q-m_M} \cdots J_{-\frac{r}{2s}+Q-m_1} |\, Q \,\rangle$$

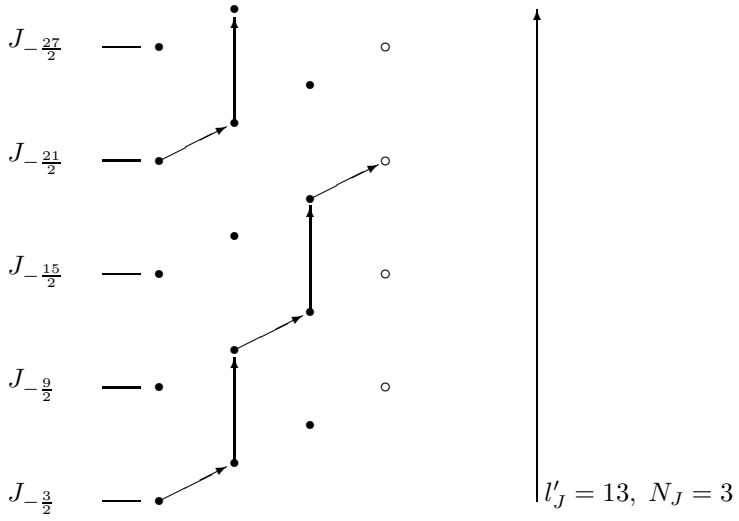

Figure 2.3: Three edge electron state, $J_{-\frac{23}{2}}J_{-\frac{13}{2}}J_{-\frac{5}{2}}|Q=0\rangle$ with $\nu = \frac{1}{3}$

$$\text{with} \quad m_M \geq \ldots \geq m_1 \geq 0. \tag{2.13}$$

Nayak and Wilczek [61] showed that there is duality between g and g^{-1} for systems satisfying Haldane-Wu state counting. We will use the same pictorial representation for edge-electron states and quasi-holes states to anticipate a stronger form of this duality. The only change will be that we interchange the roles of up-arrows and right-arrows. The edge electrons are now represented by up-arrows and the unused mode-indices by right-arrows. The labeling of the points has to be adjusted to represent the correct mode-indices. After these changes we get a graph (see figure 2.3) similar to the graph representing a many quasi-hole state but representing an electron state.
For integer spectral shift statistics (i.e. $s = 1$) we can always identify $l' - r + 1$ with the number of accessible edge electron states J within the truncated spectrum and we always have a well defined one-particle dimension.
For the electron state counting we obtain

$$\#_{\frac{r}{s}}(N_J, l'_J) = \frac{(N_J + l'_J - (N_J)\frac{r}{s})!}{N_J!(l'_J - N_J\frac{r}{s})!} \tag{2.14}$$

The state counting for the edge-electrons can be related to the state counting for quasi-holes, it amounts to interchanging $l = l'_J + N_J\frac{r}{s}$ and N which appear symmetrically in the counting formulas and rewriting $\tilde{l} = N$ as $\tilde{l} = l'_\phi - \frac{s}{r}N_\phi$ where $N_\phi = l$,

$$\#_{\frac{r}{s}}(N_J, l'_J) = \#_{\frac{s}{r}}(l'_J - N_J\frac{r}{s}, l'_J\frac{s}{r}). \tag{2.15}$$

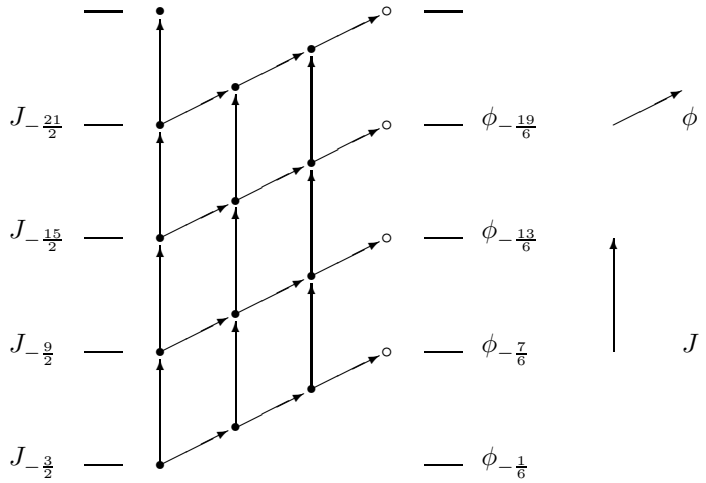

Figure 2.4: Directed graph representing all states in a truncated spectrum

The states for particles with $\frac{r}{s}$ or $\frac{s}{r}$ spectral shift statistics are in one to one correspondence with the paths on a directed graph. In figure 2.4 we have drawn an example for $s = 1,\ r = 3$ representing all the states in a truncated spectrum with $l'_J = 12$ or $l'_\phi = 4$.
Giving every path on this graph the correct Boltzmann weight we can obtain the truncated partition sum on the truncated spectrum from the directed graph. We also see there is a strong relation between $\frac{r}{s}$ and $\frac{s}{r}$ partition sums. This will turn out to be a strong form of the duality between truncated partition sums as found by [21] and is directly related to duality between the associated distribution functions. This duality will be the topic of our next section.

2.4 Truncated Partition Sums and Duality

In the preceding section we introduced a graphical representations for the states in a system satisfying spectral shift statistics. In this section we like to exploit this graphical representation to obtain stronger results on the duality properties. To illustrate the method we first rederive the duality properties found by [21] for truncated partition sums which is valid in the thermodynamic limit. Then we derive a duality relation which holds for truncated partition sums for finite size systems.
The pictures in this paragraph will refer to the $\frac{s}{r} = \frac{2}{3}$ situation and the text will describe the general case. If we represent all possible states in an mode-index interval $[m, m + l'_J]$ in one picture we get a directed graph $S(m, m + l'_J)$ on which every path denotes one of the many edge-electron states. We can do the same for the quasi-holes and obtain a directed graph $S_\phi(n, n + l'_\phi)$. If we

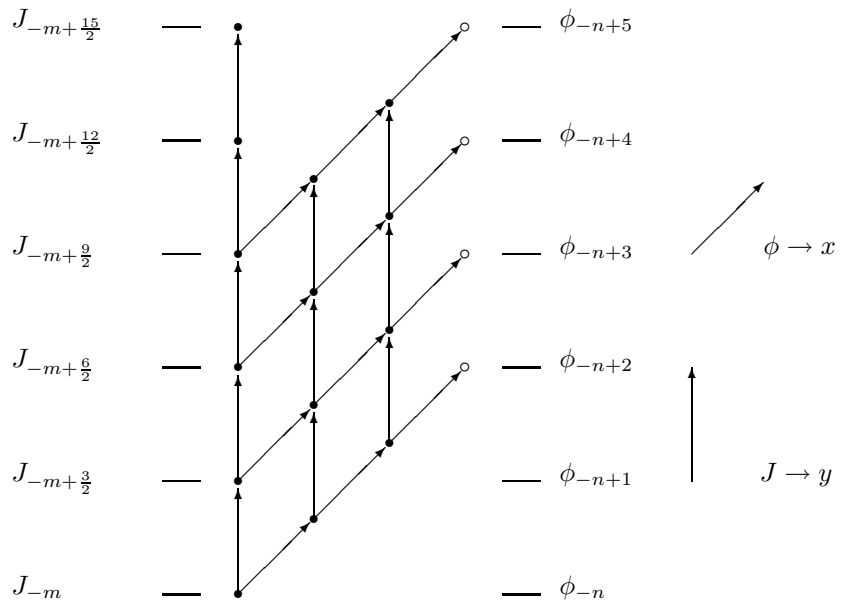

Figure 2.5: Both truncated partition sums represented in one graph.

make the choice $l'_\phi = l'_J \frac{s}{r}$ the directed graphs are the same and we can define $S(n,m,l'_J) = S_J(m, m+l'_J) = S_\phi(n, n+l'_J \frac{s}{r})$.
If we associate with every path P on the directed graph $S(n,m,l'_J)$ a weight,

$$W(P) = y^{\#(\uparrow \in P)} x^{\#(\nearrow \in P)} \tag{2.16}$$

where # denotes the number operator. Using these weights we can construct a polynomial

$$Z_{l'_J}(x,y) = \sum_{P \in S(n,m,l'_J)} W(P). \tag{2.17}$$

At this stage the notation may seem a bit overdone but we will see we will need it later; for now n and m are free parameters.
If the J or ϕ-modes under consideration have (approximately) degenerate energies, as we would have for example for anyons in the lowest Landau level or for conformal field theories in the thermodynamic limit, we can extract the truncated partition sums $X_{l'_\phi = l' \frac{s}{r}}(x)$ and $Y_{l'}(y)$ from $Z_{l'}$

$$\begin{aligned} X_{l'_J \frac{s}{r}}(x) &= Z_{l'_J}(x,1) \\ Y_{l'_J}(y) &= Z_{l'_J}(1,y). \end{aligned} \tag{2.18}$$

All the paths in a truncated spectrum have the same height and from the graph we see that we can trade r $\nearrow$ in a path for s $\uparrow$ and obtain a path in the same truncated spectrum. We express the height l'_J in the numbers of holes and the number of edge electrons

$$l'_J = \#(\nearrow) + \frac{r}{s}\#(\uparrow)$$

$$l'_\phi = \frac{s}{r}\#(\nearrow) + \#(\uparrow) \tag{2.19}$$

This is the first step in obtaining duality, if we set $y = x^{\frac{r}{s}}$ all the paths acquire the same weight $x^{l'_J}$. This property enables us to transform a weight with $x = 1$ and $y = \tilde{y}$ into a weight with $x = \tilde{x}$ and $y = 1$. Making the substitution $\tilde{y} = \tilde{x}^{-\frac{r}{s}}$ in the weight $W(P)(1, \tilde{y})$ and multiplying with $\tilde{x}^{l'_J}$ we get the weight $W(P)(\tilde{x}, 1)$. If we perform this transformation on the truncated partition sum we obtain the duality relation

$$Z_{l'_J}(1, x^{-\frac{r}{s}})x^{l'_J} = Z_{l'_J}(x, 1) \tag{2.20}$$

which is equivalent to

$$Y_{l'_J}(x^{-\frac{r}{s}})x^{l'_J} = X_{l'_J\frac{s}{r}}(x) \tag{2.21}$$

This is the same duality as we shall derive in the next chapter and leads directly to the famous duality relation between distribution functions

$$\frac{s}{r}n_{\frac{s}{r}}(x) = 1 - \frac{r}{s}n_{\frac{r}{s}}(y = x^{-\frac{r}{s}}). \tag{2.22}$$

If the separation between the energies corresponding to the mode-indices is much smaller then k_BT we can consider the derivation above as a good approximation and there is no need for more general derivation. Now we like to discuss a generalization of the duality relation 2.21 to situations where the mode-index corresponds to the energy of the quasi-particles and the separation between the energy levels is not negligible compared to k_BT.
We start with defining new weights and we have to take into account the position of a mode operator in the truncated spectrum, this is why we introduced m and n in figure 2.5. With the arrow associated with a mode operator J_{-m} we associate a weight yq_y^m and with the arrow associated with the mode-operator ϕ_{-n} we associate a weight xq_x^n. Although $q_y = q_x$ we label them differently because this enables us to obtain the partition sums $X_{(n,l'_\phi)}(x, q)$ and $Y_{(m,l'_J)}(y, q)$ from a more general polynomial associated with the directed graph by putting some of the variables to one.
The general polynomial $Z_{(n,m,l'_J)}(x, y, q_x, q_y)$ is obtained from the directed graph $S(n, m, l'_J)$ by first calculating for every path the weight by multiplying all the weights of the mode-operators corresponding to the arrows and then summing over all paths. From this construction it is clear that

$$Z_{(n,m,l'_J)}(1, y, 1, q) = Y_{(m,m+l'_J)}(y, q)$$

$$Z_{(n,m,l'_J)}(x, 1, q, 1) = X_{(n,\frac{s}{r}l'_J)}(x, q) \tag{2.23}$$

The construction of invariants is a bit more complicated and to obtain them we study the two basic operations we can do to transform one path in another.

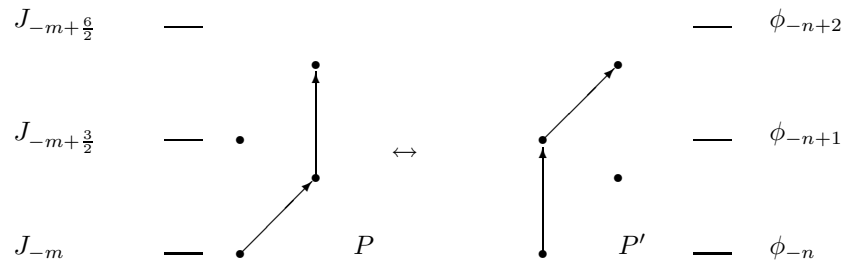

Figure 2.6: Operation I: Interchanging edge-electrons and quasi-holes

The first operation which we have drawn in figure 2.6 amounts to interchanging the order of an $\uparrow$ and a $\nearrow$ to change a path P into a path P' and we can relate the corresponding weights,

$$W(P) = W(P')q_y^{-1}q_x^1 \tag{2.24}$$

and we see that if we put $q_y = q_x$ the weight is unchanged by this transformation.

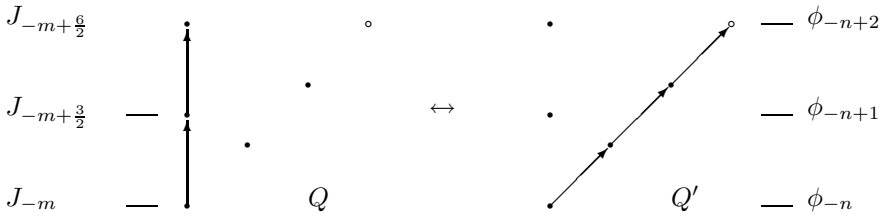

Figure 2.7: Operation II:Trading s edge-electrons for r quasi-holes

The second operation which we have drawn in figure 2.7 is to trade s $\uparrow$'s for r $\nearrow$'s. In this case we can relate the weights of the paths by

$$W(Q) = W(Q')x^{-r}y^s q_x^{-rn+s(r-1)/2} q_y^{sm+r(s-1)/2}. \tag{2.25}$$

If we like the weights to be invariant under the first operation we have to set $q_x = q_y = q$ and we obtain

$$W(Q) = W(Q')x^{-r}y^s q^{sm-rn+(s-r)/2} \tag{2.26}$$

and if we want invariance under the second operation as well we have to set y to

$$y = x^{\frac{r}{s}} q^{-m+\frac{r}{s}n-\frac{1}{2}+\frac{r}{2s}}. \tag{2.27}$$

Combining the two operations and applying them repeatedly on one path we can construct all the paths in the corresponding truncated spectrum. Because we constructed an invariant we can as before use this invariant to relate the truncated partition sum for edge electrons associated with the directed graph to the truncated partition sum for quasi-holes associated with the same graph. And we get

$$X_{(n,l'_J\frac{s}{r})}(x,q) = Y_{(m,l'_J)}(x^{-\frac{r}{s}}q^{m-\frac{r}{s}n+\frac{1}{2}-\frac{r}{2s}}, q^{-1})x^{l'_J}q^{l'_J n+l'_J(l'_J-1)/2} \tag{2.28}$$

If we set $q = 1$ we reobtain the duality relation 2.21 we found before. We use the same procedure to obtain

$$Y_{(m,l'_\phi\frac{r}{s})}(y,q) = X_{(n,l'_\phi)}(y^{-\frac{s}{r}}q^{n-\frac{s}{r}m+\frac{1}{2}-\frac{s}{2r}}, q^{-1})y^{l'_\phi}q^{l'_\phi m+l'_\phi(l'_\phi-1)/2} \tag{2.29}$$

2.5 Mutual Statistics

This section is a bit speculative in nature. First we will analyze how Haldane-Wu state counting is obtained for $su(2)_1$-spinons. Then we will speculate how spectral shift statistics can be generalized to describe the mutual statistics in Haldane-Wu state counting. Whether this corresponds to the physics in some model remains an open question.
We will argue in the next chapter that the the truncated partition sums in the $U(1)$-CFT we encountered before satisfy the recursion relation,

$$P_l = P_{l-1} + xP_{l-g}, \tag{2.30}$$

where we can read $P_l = X_l$ for $g < 1$ and $P_l = Y_l$ for $g > 1$. If the truncated partition sums are fully specified by a single recursion relation we will say that the truncated partition sums are brought in closed form.
The most naive generalization of spectral shift statistics would be to allow for two types of particles (say blue and red) which live in the same spectra and shift each others mode-indices like they shift their own. So for a quasi-particle state we would have to choose a color every time we inserted a quasi-particle. From such a formalism we obtain the following counting formula

$$\#(N_{blue}, N_{red}, l) = \frac{(N_{blue}+N_{red}+l)!}{N_{blue}!N_{red}!l!}. \tag{2.31}$$

which is clearly different from Haldane-Wu state counting.
It is possible to introduce mutual statistics into spectral shift statistics, although we will need an extra principle to get the state counting right as is clear from the example above. To get some insight we will first discuss how to get theories with statistics matrix G satisfying

$$g_{ij} = g_{ii} \tag{2.32}$$

for which the truncated partition sums can be shown to have a closed form. Secondly we will describe how the more complicated structure needed for the more

general statistics matrices [11] can be obtained in a spectral shift description. The more complicated structure is in general reflected in the need for more then one type of truncated partition sum [71, 14].
If we interpret Haldane-Wu state counting for particles living in the same truncated spectrum it turns out that it corresponds to saying that for the the counting the only important property is how the particles of every species are positioned with respect to the holes. Within a set of particles, however, the ordering is unimportant. In $su(2)_1$ conformal field theories this property is a result of the identification of two states if they only differ by a permutation of two particles at neighboring mode-indices. This introduces a symmetry under the generators of the permutation group on a cluster of particles without holes and thus identifies all permutations of the particles in a cluster with one state. So for state counting purposes all permutations inside a cluster give back the same state and a cluster is specified if for every species the number of particles in the cluster is specified. To make this mechanism work properly the shifts introduced by a pair of particles belonging to different species should be independent of the order in which they where introduced in a cluster. The simplest condition implementing this condition is if the shift G_{ij} introduced by a species j particle is independent of i.
An example are the $su(2)_1$ spinons which have have statistics matrix

$$G = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{2.33}$$

and the pairing is clearly visible as a subtraction in the recursion relation for truncated spectra,

$$P_l = (1 - x_1 x_2) P_{l-1} + (x_1 + x_2) P_{l-\frac{1}{2}} \tag{2.34}$$

where x_1 and x_2 are the fugacities of the two types of spinons. Every hole, the 1 multiplying P_{l-1} in the recursion relation, is accompanied by a subtraction $-x_1 x_2$, which identifies two states which differ only by a permutation of two neighboring particles and were created simultaneously with the hole.
This construction can be generalized to the quasi-particles satisfying $g_{ij} = g_{ii}$ for which we propose the recursion relation

$$P_l = P_{l-1} + \sum_i x_i P_{l-g_{ii}} - \sum_{j<k} x_j x_k P_{l-g_{jj}-g_{kk}}. \tag{2.35}$$

The subtraction here plays the same role as before and this recursion relation will therefore give rise to Haldane-Wu state counting.
For more complicated variations on spectral shift statistics the correlations in the spectrum cannot be described by local rules, i.e. subtractions in the recursion matrix, and to keep track of non-local correlations the truncated partition sum is split in several parts. For these parts a recursion matrix can be defined but the largest eigenvalues of this recursion matrix no longer correspond to approximate one-level partition sums.

Recently an other scheme was proposed based on finitized partition sums [11] where to every species there is associated a separate set of quantum numbers which brings us closer to the original state counting formulas where the 'dimensions' for different particles can be varied independently,

$$P_{l_1,\ldots,l_i,\ldots,l_N} = P_{l_1,\ldots,l_i-1,\ldots,l_N} + x_i P_{l_1-g_{1i},\ldots,l_i-g_{ii},\ldots,l_N-g_{Ni}} \tag{2.36}$$

where for every point we have N recursion relations. To construct $P_{l_1,\ldots,l_i,\ldots,l_N}$ from the other finitized partition sums we have to use a linear combination of these recursion relations. In other words, if we would represent the finitized partition sums by points in a N-dimensional lattice we can for every point specify two arrows connecting it to two other lattice points, in such a way that these arrow points to finitized partition sums appearing in one of the N recursion relations. In general if we repeat this procedure we will get more and more paths from more and more finitized partition sums which contribute to one finitized partition sum. The case $g_{ij} = g_{ii}$ is special because all the finitized partition sums contributing to the history of a partition sum are on one line, because of this their recursion relations are closed. In some cases the partition sums are not on a line but the recursion relations can be chosen such that all finitized partition sums contributing to the history fall in a small tube for this case it can happen that a closed form of the recursion relations is possible.
The recursion relation discussed above suggest the following generalization of spectral shift statistics to a N species situation, if a particle of species i is placed the quantum numbers accessible to the next particle will be

$$\epsilon^j_{n_1,\ldots,n_i+1,0,\ldots,0} = \epsilon^j_{n_1,\ldots,n_i,0,\ldots,0} + g_{ji} + k^i_{n_i}\delta_{ji}$$

$$\text{with } n_i > 0 \text{ and}$$

$$\epsilon^j_{n_1,\ldots,1,0,\ldots,0} = \epsilon^j_{n_1,\ldots,n_i,0,\ldots,0} + g_{ji} + k^i_{n_i}\delta_{ji} + k^{i-1}_{n_{i-1}+1}\delta_{j,i-1}$$

$$\text{with } n_i = 0. \tag{2.37}$$

Note that to get Haldane-Wu state counting we had to specify the order in which to place the species, which might be equivalent to specifying an order in the application of the recursion relations. If the statistics matrix is symmetric, $g_{ij} = g_{ji}$, generalized spectral shift statistics will lead to the same finitized partition sums as were found from the recursion relations above [11]. For non-symmetric statistics matrices the finitized partition sum would depend on the order in which the species are added an interpretation of such a case in terms of spectral shift statistics would therefore be unclear if no extra information is available.
This generalized spectral shift statistics description explicitly constructs a set of basis states by specifying when particles maybe added to a state. The recursion relation method explicitly constructs finitized partition sums by looking back at the finitized partition sums already constructed. An open problem is whether these two approaches can be explicitly connected.

2.6 Spectral Shift vs. Generalized Exclusion

The history of fractional statistics shows that the term generalized exclusion statistics has often been used once Haldane-Wu type of state counting was established. We propose to use a more refined terminology which makes a clear distinction between what we call (generalized) spectral shift statistics , Haldane's generalized exclusion statistics and other types of statistics which do not fit onto the scheme of exclusion or spectral shift statistics. An example of the latter would be the statistics of Gentile parafermions.
Haldane formulated two properties of generalized exclusion statistics which we rephrased in section 2.1 using the terminology of Ilinski et al. [37]. It is clear that spectral shift statistics does not satisfy the state homogeneity property. We found that in conformal field theories this property is violated because spectral shift statistics is found there and particle homogeneity is only valid in $U(1)$ conformal field theories if we reinterpret it as a statement about the spectral shift parameter. Ilinski et al. pointed out that in general TBA systems do not satisfy particle and state homogeneity. These results from conformal field theory and TBA show that different counting schemes occur which are more complex than exclusion statistics or the (generalized) spectral shift statistics proposed here, which nevertheless lead to the same state counting.
Exclusion statistics was originally defined to generalize the Pauli-principle such that it would give Haldane-Wu state counting. It is hard to think of a plausible mechanism giving true fractional dimensions and it is our belief that statistics satisfying Haldane's original definition will always be integer and will come with a second quantized formulation. The states in such a exclusion statistics system would be labeled by a set of quantum numbers. One of these quantum numbers would live in a space with the same dimension as the exclusion statistics parameter. when the other quantum numbers are kept fixed only one of the quantum numbers in this space can be occupied.
A possible realization of integer exclusion statistics would be a molecule with two binding sites where another molecule can bind, but to which no second molecule can bind. A possible mechanism for this scenario would be that the two binding sites are so close that once one is occupied the other site is blocked. Another possible realization is found in the $t-J$ model. This model has a spin degree of freedom on every site but strong on site repulsion makes it impossible to occupy it more then once, so we get $g=2$ exclusion statistics. The important difference between these models and the statistics in conformal field theories is that in conformal fields theories there is correlation between all mode-indices. In exclusion statistics there is only a correlation between the occupied state and the $g-1$ excluded states and the distribution function directly gives the probability of finding a particle with a given energy.
There seems to be room for many other notions of statistics in the context of strongly correlated systems, the strongest methods so far which seem to be capable of describing all statistical interactions in 1D-systems are those of finitized or truncated partition sums [71, 15, 14, 50] which can be used to describe such various statistics as generalized exclusion statistics, spectral shift statistics, gentile

parafermion statistics, non-Abelian statistics. We will introduce the truncated partition sums in the next chapter when we derive the distribution functions for the collective quantum numbers of $U(1)$-conformal field theories ($U(1)$-CFT).

3

Quasi-Particle Thermodynamics

In the preceding chapter we discussed the statistics of the quasi-particles describing a $U(1)$-conformal field theory associated with a fqH-edge. In this chapter we will again study this basis, which we will call the pre-fqH basis, but now we will use the method of truncated or finitized chiral partition sums. This method was proposed by Schoutens in reference [71] and here it leads to 1-particle distribution functions which satisfy the Isakov-Ouvry-Wu (IOW) equations. We will interpret these thermodynamic results as properties of fractional quantum Hall systems.

The other examples discussed in [71] include spinons in the $\widehat{su(n)}_1$ WZW models and CFT parafermions. The $\widehat{su(n)}_k$ Wess-Zumino-Witten models [32, 16, 12, 15] possess so-called spinon bases, which are analogous to the fqH quasi-particle basis. The $\hat{su(n)}_k$ spinons and the CFT parafermions do not satisfy Haldane-Wu state counting in its simplest form. Nevertheless the method used in this chapter can be used to derive their thermodynamic properties. This method is therefore more general applicable then the spectral shift statistics presented in the last chapter.

In this chapter we will first present the method of truncated partition sums. For this purpose we will once more give the basis we discussed in detail in the last chapter and show this basis leads to simple recursion relations between the truncated partition sums. Then we show how these recursion relations can be used to derive the distributions functions and the Isakov-Ouvry-Wu equations [42]. We give another proof of the duality property we discussed in the last chapter. Furthermore, we add a short section on composite edges. And we conclude with a section in which we calculate the specific heat of the fractional quantum Hall edge and the Hall conductance. These calculations show that the method reproduces the correct physics. Furthermore, it shows that both fractional charge and fractional statistics are needed to get the correct Hall conductance.

3.1 Quasi-Particles for the $\nu = \frac{1}{p}$ fqHe edge

We consider the finite size spectrum for the CFT describing a single $\nu = \frac{1}{p}$ fqHe edge. In the CFT jargon, this theory is characterized as a $c = 1$ chiral free boson theory at radius $R^2 = p$. We shall consider the chiral Hilbert space corresponding to the following partition function

$$Z^{1/p}(q) = \sum_{Q=-\infty}^{\infty} \frac{q^{\frac{Q^2}{2p}}}{(q)_\infty} \ , \tag{3.1}$$

with $(q)_\infty = \prod_{l=1}^{\infty}(1-q^l)$ and $q = e^{-\beta \frac{2\pi}{L}\frac{1}{\rho_0}}$. The 1-particle energies are of the form $\epsilon_l = l\frac{2\pi}{L}\frac{1}{\rho_0}$ with l integer and ρ_0 the density of states per unit length, $\rho_0 = (\hbar v_F)^{-1}$. In this formula, the $U(1)$ affine Kac-Moody symmetry is clearly visible as all states at fixed $U(1)$ charge Q form an irreducible representation of this symmetry.
We should stress that the Hilbert space corresponding to eq. 3.1 is not the physical Hilbert space for the edge theory of a fractional quantum Hall sample with the topology of a disc. In the latter Hilbert space physical charge is quantized in units of e and, correspondingly, the $U(1)$ charge Q in eq. 3.1 is restricted to multiples of p [81]. In the geometry of a Corbino disc, i.e. a disc with a hole at the centre, the operator that transfers charge $\frac{e}{p}$ from the outer edge to the inner edge is physical. Accordingly, the physical Hilbert space is obtained by taking a tensor product of left and right copies of the Hilbert space eq. 3.1 and restricting the total $U(1)$ charge $Q_L + Q_R$ to multiples of p [81]. In the quasi-particle formalism that we present below the various restrictions on Q_L, Q_R are easily implemented.
Our goal here is to understand the collection of states in eq. 3.1 in a different manner, and to view them as multi- particle states built from the creation operators for edge quasi-particles J and and quasi-holes ϕ. Due to the above-mentioned restrictions on the $U(1)$ charges Q and Q_L+Q_R, the chiral quasi-hole operator $\phi(z)$ by itself is not a physical operator in edge theories for the disc or cylinder. Physical states are obtained by restricting the number of ϕ-quanta in the appropriate manner.

3.1.1 Quasi-Hole States

We start by considering quasi-hole states that are built by applying only the modes ϕ_{-s} defined via $\phi(z) = \sum_s \phi_{-s} z^{s-\frac{1}{2p}}$. Clearly, the index s gives the dimensionless energy of the mode ϕ_{-s}. When acting on the charge-0 vacuum $|0\rangle$, we find the following multi-ϕ states (compare with [12] for the case $p = 2$, see also [43])

$$\phi_{-\frac{(2N-1)}{2p}-n_N} \cdots \phi_{-\frac{3}{2p}-n_2}\phi_{-\frac{1}{2p}-n_1}|\,0\,\rangle$$

with

$$n_N \geq n_{N-1} \geq \ldots \geq n_1 \geq 0 \ . \tag{3.2}$$

The choice of minimal modes is such that the lowest state of charge $Q\frac{e}{p}$ is at energy $\frac{Q^2}{2p}$, in agreement with the scaling dimension of the corresponding CFT primary field. Using so-called generalized commutation relations satisfied by the modes ϕ_{-s} one may show [12] that all multi-ϕ states different from those in the list 3.2 are either zero or linear combinations of states from the list 3.2.
Before writing more general states we shall first focus on the spectral shift statistics properties of the quanta ϕ_{-s}. We follow the procedure of [71] and start by introducing truncated partition sums for the quasi-hole states 3.2. For $s = \frac{1}{2p}$, $\frac{3}{2p}$, etc, we define polynomials $P_s(x,q)$ to keep track of the number of many-body states that can be made using only the modes ϕ_{-k} with $k \leq s$, and that have a highest occupied mode with energy s' such that $s - s'$ is integer. $P_s(x,q)$ is defined as the trace of the quantity $x^N q^E$ over all these states, where N is the number of quasi-holes, E is the dimensionless total energy, and $x = e^{\beta\mu_{\rm qh}}$. For $p = 3$ this gives

$$P_{\frac{1}{6}} = xq^{\frac{1}{6}} \ , \quad P_{\frac{1}{2}} = x^2 q^{\frac{4}{6}} \ , \quad P_{\frac{5}{6}} = 1 + x^3 q^{\frac{9}{6}} \ , \quad \text{etc.} \tag{3.3}$$

In general, a occupied quasi-hole state of energy s corresponds to a factor xq^s in these generating polynomials.
The systematics of the edge quasi-hole states 3.2 directly lead to the following recursion relations between the polynomials $P_s(x,q)$,

$$P_s(x,q) = P_{s-1}(x,q) + x\, q^s\, P_{s-\frac{1}{p}}(x,q) \ . \tag{3.4}$$

For $p = 1$, which is the case corresponding to a $\nu = 1$ integer qHe edge, this relation directly implies $P_{l-\frac{1}{2}}(x,q) = \prod_{j=1}^{l}(1 + xq^{j-\frac{1}{2}})$. In that case the partition sum is simply a product and we recognize free fermions. For general m things are not that simple, but we can rewrite the recursion relation in matrix form

$$\begin{pmatrix} P_{l-\frac{2p-1}{2p}} \\ \vdots \\ P_{l-\frac{1}{2p}} \end{pmatrix} = M_l^{\rm qh}(x,q) \begin{pmatrix} P_{l-1-\frac{2p-1}{2p}} \\ \vdots \\ P_{l-1-\frac{1}{2p}} \end{pmatrix}, \tag{3.5}$$

with $l = 1, 2, \ldots$ and $M_l^{\rm qh}(x,q)$ the following $p \times p$ matrix

$$M_l^{\rm qh}(x,q) =$$

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & xq^{l-\frac{2p-1}{2p}} \\ xq^{l-\frac{2p-3}{2p}} & 1 & \ldots & 0 & x^2 q^{2l-\frac{4p-4}{2p}} \\ \vdots & & \ddots & & \vdots \\ x^{p-1}q^{(p-1)l-\frac{p}{2}+\frac{2p-1}{2p}} & x^{p-2}q^{(p-2)l-\frac{p}{2}+\frac{4p-4}{2p}} & \ldots & xq^{l-\frac{1}{2p}} & 1 + x^p q^{pl-\frac{p}{2}} \end{pmatrix}. \tag{3.6}$$

The grand partition function for the quasi-hole states 3.2 is then given by

$$Z^{\rm qh}(x,q) = (\,1 \quad 1 \quad \dots \quad 1\,) \left(\prod_{l=1}^{\infty} M_l^{\rm qh}(x,q)\right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} . \tag{3.7}$$

We propose that the quasi-hole modes ϕ_{-s} with $s = l - \frac{2p-1}{2p}, \dots, l - \frac{1}{2p}$ be viewed as a single (p-fold degenerate) level in the collective coordinate spectrum. This convention is natural since a single quasi-particle over the ground state can only occupy one of these p levels. The $p \times p$ matrix $M_l^{\rm qh}$ is then a level-to-level transfer matrix and replaces the free fermion ($p = 1$) factor $(1 + xq^{l-\frac{1}{2}})$. We assume that the thermodynamics of the states 3.2 will be dominated by the largest eigenvalues $\lambda_l^+(x,q)$ of the matrices $M_l^{\rm qh}(x,q)$. These satisfy the characteristic equations

$$(\lambda_l^+ - 1)^p - x^p q^{pl-\frac{p}{2}} (\lambda_l^+)^{p-1} = 0 \ . \tag{3.8}$$

Instead of trying to solve these equations, we can derive from them a result for the 1-particle distribution functions

$$\bar{n}^{\rm qh}(l) \equiv x\partial_x \ln(\lambda_l^+) = x\frac{\partial_x \lambda_l^+}{\lambda_l^+} \ . \tag{3.9}$$

We find

$$\bar{n}^{\rm qh}(l) = \frac{\lambda_l^+ - 1}{1 + \frac{1}{p}(\lambda_l^+ - 1)},$$

$$(xq^{l-\frac{1}{2}})^{-1} = (\lambda_l^+ - 1)^{-1} (\lambda_l^+)^{1-\frac{1}{p}} \ . \tag{3.10}$$

Comparing with 2.2 and identifying $g = \frac{1}{p}$ and $w(\epsilon) = (\lambda_l^+ - 1)^{-1}$, we see that the distribution function $\bar{n}^{\rm qh}(l)$ becomes identical to $\bar{n}_{g=\frac{1}{p}}(\epsilon = l)$. In other words, the statistics properties of the $\nu = \frac{1}{p}$ quasi-holes are those associated with Haldane-Wu state counting, with $g = \frac{1}{p}$. This identification is consistent with the result of bosonization applied to g-ons [84], with the character computations of [36], and with our observations in the previous chapter.
For the case $p = 2$, which is not in the category of fqHe edges, the equilibrium distribution is given by

$$\bar{n}_{\frac{1}{2}}(\epsilon) = \frac{2}{\sqrt{1 + 4e^{-2\beta(\mu-\epsilon)}}} \ . \tag{3.11}$$

For $p = 3$ the explicit formulas (obtained using the Cardano formula for cubic equations) are quite unpleasant; figure 3.1 shows the distribution $\bar{n}_{\frac{1}{3}}(\epsilon)$.

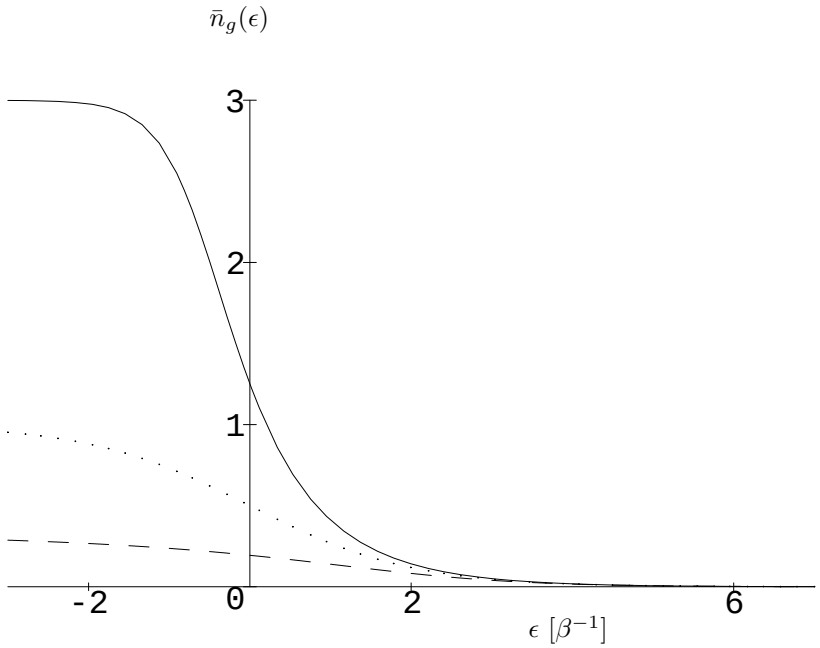

Figure 3.1: Distribution functions for fractional exclusion statistics with $g = 3$ (dashed line), $g = 1$ (dotted line), and $g = \frac{1}{3}$ (solid line) , all at the same temperature and at zero chemical potential.

3.1.2 Edge Electron States

The same procedure can be applied to the edge electrons, which are created by modes J_{-t} with $J(z) = \sum_t J_{-t} z^{t-\frac{p}{2}}$. Multi-electron states take the form

$$J_{-(2M-1)\frac{p}{2}-m_M} \cdots J_{-3\frac{p}{2}-m_2} J_{-\frac{p}{2}-m_1} |\, 0 \,\rangle$$

with

$$m_M \geq m_{M-1} \geq \ldots \geq m_1 \geq 0 \tag{3.12}$$

and we have truncated partition sums $Q_t(y,q)$ with t a half-odd-integer and $y = e^{\beta\mu_e}$. They satisfy the recursion relations

$$Q_t(y,q) = Q_{t-1}(y,q) + y\, q^t\, Q_{t-p}(y,q), \tag{3.13}$$

with the following initial values

$$Q_{-\frac{p}{2}} = \ldots = Q_{\frac{p}{2}-1} = 1\ . \tag{3.14}$$

The 'transfer matrix' for the edge electrons $M^{\rm e}_k(y,q)$ is defined by

$$\begin{pmatrix} Q_K \\ \vdots \\ Q_{K+p-1} \end{pmatrix} = M^{\rm e}_k(y,q) \begin{pmatrix} Q_{K-p} \\ \vdots \\ Q_{K-1} \end{pmatrix} \tag{3.15}$$

with $K = kp - p/2$, $k = 1, 2, \ldots$ and we have

$$Z^{\rm e}(y,q) = (\,1 \quad 0 \ldots \quad 0 \quad 0\,) \left(\prod_{k=1}^{\infty} M^{\rm e}_k(y,q) \right) \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}. \tag{3.16}$$

In this case, a single action of the transfer matrix comprises a jump of p 1-particle levels, and the relevant distribution function will be

$$\bar{n}^{\rm e}(k) \equiv y\partial_y \ln[(\mu^+_k)^{\frac{1}{p}}] = \frac{y}{p}\frac{\partial_y \mu^+_k}{\mu^+_k}, \tag{3.17}$$

The characteristic equation for the eigenvalue μ^+_k

$$\prod_{i=0}^{p-1} (\mu^+_k - yq^{pk-i}) - (\mu^+_k)^{p-1} = 0, \tag{3.18}$$

leads to

$$\bar{n}^{\rm e}(k) = \frac{1}{p + (h_k - 1)}\ , \quad (yq^{pk})^{-1} = (h_k - 1)^p h_k^{1-p}. \tag{3.19}$$

with $h_k = \mu^+_k/(yq^{pk})$. Identifying $w(\epsilon) = h_k - 1$, we again recognize the IOW equations 2.2 for Haldane-Wu state counting, this time with $g = p$, and we may identify $\bar{n}^{\rm e}(k)$ with $\bar{n}_{g=p}(\epsilon = pk)$.
For $p = 2$ this gives

$$\bar{n}_2(\epsilon) = \tfrac{1}{2}\left(1 - \frac{1}{\sqrt{1 + 4\,e^{-\beta(\epsilon-\mu)}}}\right)\ . \tag{3.20}$$

See figure 3.1 for the distribution function $\bar{n}_3(\epsilon)$ at $\mu_{\rm e} = 0$.

3.1.3 Duality

Having recognized distribution functions for Haldane-Wu state counting with $g = \frac{1}{p}$ and $g = p$, respectively, we expect a particle-hole duality between the two cases (compare with [61, 66] and section 2.4).
Before we come to that, we generalize the results of the preceding sections by considering a chiral $c_{CFT} = 1$ CFT of compactification radius $R^2 = r/s$, with $r > s$ and r, s coprime. Choosing ϕ-quanta of charge $+\frac{s}{r}e$ and J-quanta of charge

$-e$ as our fundamental excitations, we easily repeat the previous analysis and derive the following recursion relations

$$X_l(x) = X_{l-r}(x) + x\, X_{l-s}(x) \ , \qquad Y_l(y) = Y_{l-s}(y) + y\, Y_{l-r}(y) \ , \tag{3.21}$$

where we put $q = 1$ for convenience. The connection with the quantities P_l and Q_l defined for $\frac{r}{s} = p$ is $X_l \leftrightarrow P_{\frac{2l-1}{2p}}$, $Y_l \leftrightarrow Q_{\frac{p}{2}+(l-1)}$. Proceeding as before we obtain the distribution functions for Haldane statistics with $g = s/r$ (for the ϕ-quanta) and $\tilde{g} = r/s$ (for the J-quanta).
In the papers [61, 66] it was recognized that the cases with g and $\tilde{g} = 1/g$ are dual in the sense that particles are dual to holes. To recover this duality in our present approach, we note that if $Y_l(y)$ is a solution of the second relation in 3.21, the expression

$$X_l(x) = Y_l(y = x^{-\tilde{g}})\, x^{\frac{l}{s}} \tag{3.22}$$

solves the first recursion relation. Assuming $r > s$, we can rewrite both recursion relations in a form involving a $r \times r$ recursion matrix. The largest eigenvalues $\lambda^+(x)$ and $\mu^+(y)$ are then related via

$$\lambda^+(x) = \mu^+(x^{-\tilde{g}})\, x^{\tilde{g}} \tag{3.23}$$

and the distribution functions

$$\bar{n}_g(x) \equiv x\partial_x \ln \lambda^+(x) \ , \qquad \bar{n}_{\tilde{g}}(y) \equiv g\, y\partial_y \ln \mu^+(y) \ , \tag{3.24}$$

satisfy

$$g\, \bar{n}_g(x) = 1 - \tilde{g}\, \bar{n}_{\tilde{g}}(y = x^{-\tilde{g}}) \ , \tag{3.25}$$

or, putting $\mu_{\tilde{g}} = -\tilde{g}\mu_g$ and restoring $q \neq 1$

$$g\, \bar{n}_g(\epsilon) = 1 - \tilde{g}\, \bar{n}_{\tilde{g}}(-\tilde{g}\epsilon) \ , \tag{3.26}$$

in agreement with the results of [61]. The interpretation of this result is that the $\tilde{g}$ quanta with positive energy act as holes in the ground state distribution of negative energy g-quanta. The relative factor $(-\tilde{g})$ between the energy arguments in eq. 3.26 indicates that the act of taking out r g-quanta corresponds to adding s $\tilde{g}$-quanta. This duality further implies that, when setting up a quasi-particle description for fqHe edges, we can opt for a pre-CS basis with either quasi-holes òr edge electrons, with energies over the full range $-\infty < \epsilon < \infty$, or a pre-fqH basis with a combination of both types of quasi-particles, each having positive energies only. The CFT finite size spectrum naturally leads to to a pre-fqH basis (see the next section), while the analogy with Calogero-Sutherland quantum mechanics naturally leads to a pre-CS basis, see ch. 4.

3.1.4 The Full Spectrum

To complete our quasi-particle description for the $\nu = \frac{1}{p}$ edge, we need to specify how quasi-hole and electron operators can be combined to produce a complete

basis for the chiral Hilbert space 3.1. We consider the following set of states, the pre-fqH basis,

$$J_{-(2M-1)\frac{p}{2}+Q-m_M}\cdots J_{-\frac{p}{2}+Q-m_1}\phi_{-(2N-1)\frac{1}{2p}-\frac{Q}{p}-n_N}\cdots\phi_{-\frac{1}{2p}-\frac{Q}{p}-n_1}|\,Q\,\rangle$$
$$\text{with}\quad m_M\geq m_{M-1}\geq\ldots\geq m_1\geq 0,\quad n_N\geq n_{N-1}\geq\ldots\geq n_1\geq 0\,,$$
$$n_1>0\quad\text{if}\quad Q<0\,, \tag{3.27}$$

where $|Q\rangle$ denotes the lowest energy state of charge $Q\frac{e}{p}$ with Q taking the values $-(p-1)$, $-(p-2)$, …, -1, 0. Our claim is now that the collection 3.27 forms a basis of the chiral Hilbert space, so that

$$Z^{1/p}(q)=\sum_{Q=-(p-1)}^{0}q^{\frac{Q^2}{2p}}\,Z_Q^{\rm qh}(x=1,q)\,Z_Q^{\rm e}(y=1,q) \tag{3.28}$$

where we added a factor $q^{\frac{Q^2}{2p}}$ to take into account the energy of the initial states and we denoted by $Z_Q^{\rm qh}$ and $Z_Q^{\rm e}$ the generalizations of the partition functions 3.7 and 3.16 to the sector with vacuum charge Q. They are naturally written as

$$Z_Q^{\rm qh}(x,q)=\sum_{N=0}^{\infty}\frac{x^N q^{\frac{1}{2p}(N^2+2QN)+(1-\delta_{Q,0})N}}{(q)_N},$$
$$Z_Q^{\rm e}(y,q)=\sum_{M=0}^{\infty}\frac{y^M q^{\frac{p}{2}M^2-QM}}{(q)_M}\,, \tag{3.29}$$

with $(q)_L=\prod_{l=1}^{L}(1-q^l)$.
While the collection of states 3.27 looks rather complicated, it may be understood by considering the special case $p=1$, which is a theory of two real free fermions of charge ±1. In this case there is only the $Q=0$ vacuum and the allowed ϕ and J modes reduce to the familiar free fermion modes $\psi^{\pm}_{-\frac{1}{2}-n_j}$.
The right hand side of 3.28 has the form of a so-called 'fermionic sum formula' [50] and the equality of 3.1 and 3.28 is a new Rogers-Ramanujan identity.
Similar identities relating 'fermionic sums' to characters in conformal field theories have been studied in the literature, see for example [50, 12, 13]. We would like to stress that the reasoning leading to these identities is very different between our approach and the work of [50]: in our approach the identities express spectral shift statistics properties of CFT fields, while in the work of Kedem et al. the identities are based on Bethe Ansatz solutions of specific integrable lattice models. The first example where these two approaches have been explicitly connected is that of spinons in $SU(2)_1$ CFT and in the associated Haldane-Shastry spin chains [32, 16, 12].
The important conclusion from the above is that, up to a finite sum over vacuum charges, the chiral partition sum factorizes as a product of a quasi-hole piece

and an edge electron piece. This means that the two types of quasi-particles are independent, or, in other words, that they do not have any mutual statistics. This property will be reflected in the specific heat. When we calculate the specific heat in section 3.3 we will see that the sum of the specific heat due to only edge-electrons and only quasi-holes adds up to the correct specific heat for a $c = 1$-CFT. If we would add up the specific heats of the edge-electrons and their positive charge counterparts we would find a specific heat which is to low to describe a $c = 1$-CFT, i.e. we missed some states. If on the contrary we add up the specific heats of the quasi-holes and quasi-electrons, the sum will be larger than the actual specific heat and we overcount the number of states.
This then explains our asymmetric choice of quasi-particles. Had we chosen to work with fundamental quasi-particles of charges $\pm\frac{e}{p}$, we would have come across non- trivial mutual statistics. All of this is nicely illustrated with the case $p = 2$ where we can opt for the pre-fqH basis with independent quasi-holes and edge electrons, or for a 'spinon basis' built from charge $\pm\frac{e}{2}$ quanta, which are identical to the spinons of [32, 16, 12] and which have a non-trivial 2×2 statistics matrix. The two choices have the quasi-hole states 3.2 (called 'fully polarized spinon states' in [12]) in common, but differ in the way negative charges are brought in.

3.2 Composite Edges: Jain Series

In this section we briefly describe a quasi-particle formulation of the composite edge theories corresponding to the filling fractions $\nu = \frac{\nu*}{2m\nu*+1}$ of the Jain series. These edge theories can be written as a collection of $n = \nu^*$ free bosons, coupled via the topological K-matrix of the effective bulk Chern-Simons theory [81]. In [26] it was shown that the effective low-energy CFT for particles satisfying Haldane-Wu state counting with $n \times n$ statistical matrix G is a $c = n$ CFT with topological matrix $K = G^{-1}$. Inverting the argument we expect that the fundamental excitations of the CFT for qHe matrix K can be interpreted in terms of pseudo-particles satisfying fractional exclusion statistics with matrix $G = K^{-1}$.
An alternative and more natural approach to the Jain series edges would be to first perform a change of basis which separates a single charged mode from a set of $n-1$ neutral modes [47, 58]. The latter are governed by an $\widehat{su(n)_1}$ affine Kac-Moody symmetry, and can be treated separately. An option is to view them as a set of n free parafermions in the sense of Gentile, see [71]. The CFT for the remaining charged mode is of the type that we described in this paper, with $g = \nu$. The entire edge theory is then described by a single (charged) g-on and a set of $\widehat{su(n)}_1$ degrees of freedom.
As an example of how the chiral Hilbert space works out, here is the example of $\nu = 2/5$, with K matrix

$$K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} . \tag{3.30}$$

This theory has two independent $U(1)$ affine Kac-Moody symmetries, giving a factor $\left[\prod_{l=1}^{\infty}(1-q^l)\right]^{-2}$ in the partition function. The various charge sectors are labeled by pairs of integers (l_1, l_2), the energy being given by $E(l_1,l_2) = \frac{1}{10}(3l_1^2 - 4l_1l_2 + 3l_2^2)$ (this is the bilinear form defined by the inverse of the K-matrix 3.30). Thus

$$Z_{\nu=2/5}(q) = \sum_{(l_1,l_2)} \frac{q^{E(l_1,l_2)}}{\left[\prod_{l=1}^{\infty}(1-q^l)\right]^2}. \tag{3.31}$$

Under the rearrangement into $\widehat{su(2)}_1$ times $U(1)$, the combination $\frac{1}{2}(l_2 - l_1)$ plays the role of the $su(2)$ spin, while $l_1 + l_2$ is the charge under the new $U(1)$. The character identity will be [58]

$$Z_{\nu=2/5}(q) = \chi_{j=0}^{su(2)_1}(q) Z^{2/5}_{\rm even}(q) + \chi_{j=\frac{1}{2}}^{su(2)_1}(q) Z^{2/5}_{\rm odd}(q) \ , \tag{3.32}$$

where the subscript even (odd) on $Z^{s/r}$ means that we restrict to the states with total $U(1)$ charge Q even or odd. Simple expressions for the $su(2)_1$ characters are

$$\chi_{j=0}^{su(2)_1}(q) = \sum_{m+n \text{ even}} \frac{q^{\frac{1}{4}(m+n)^2}}{(q)_m (q)_n} \ , \qquad \chi_{j=\frac{1}{2}}^{su(2)_1}(q) = \sum_{m+n \text{ odd}} \frac{q^{\frac{1}{4}(m+n)^2}}{(q)_m (q)_n} \ . \tag{3.33}$$

For the general case with $\nu = \frac{n}{nm+1}$, the charged sector is described by a free boson CFT at compactification radius $R^2 = \nu^{-1}$, which we write as $R^2 = \frac{r}{s}$. The chiral partition sum is

$$Z^{s/r}(q) = \sum_{Q=-\infty}^{\infty} \frac{q^{Q^2/(2rs)}}{\prod_{l=1}^{\infty}(1-q^l)} \ , \tag{3.34}$$

and restrictions, such as the even/odd in eq.3.32 are taken into account by restricting the charge quantum number Q.

Our fundamental charged edge quasi-particles will now be the primary fields of $U(1)$ charges $+s$ and $-r$; we shall write the creation and annihilation modes of these fields as ϕ_{-t} and J_{-t}, respectively. Note that for $s \neq 1$ the operators J_{-t} are not the physical edge electrons as the latter can only be written by including non-trivial factors from the neutral sector!

In close analogy with our analysis in the previous section, we can now establish that the states

$$J_{-(2M-1)\frac{r}{2s}+\frac{Q}{s}-m_M} \cdots J_{-\frac{r}{2s}+\frac{Q}{s}-m_1} \phi_{-(2N-1)\frac{s}{2r}-\frac{Q}{r}-n_N} \cdots \phi_{-\frac{s}{2r}-\frac{Q}{r}-n_1} | \, Q \, \rangle$$

$$\begin{aligned} &\text{with} \qquad m_M \geq m_{M-1} \geq \ldots \geq m_1 \geq 0, \quad n_N \geq n_{N-1} \geq \ldots \geq n_1 \\ &\text{and} \qquad n_1 \geq 0 \quad \text{if } Q \geq 0 \\ &\qquad\quad\ \ n_1 > 0 \quad \text{if } Q < 0 \ , \end{aligned} \tag{3.35}$$

with $Q = -(r-s), \ldots, +(s-1)$, span the chiral Hilbert space 3.34 of the charged boson. The total energy of the lowest energy state in the charge sector Q having particle-numbers M and N for the quanta of type J and ϕ, respectively, equals

$$E(Q;M,N) = \frac{Q^2}{2rs} + \frac{r}{2s}M^2 - \frac{Q}{s}M + \frac{s}{2r}N^2 + \left[\frac{Q}{r} + \delta_{Q<0}\right] N \tag{3.36}$$

and this leads to the following expression for the chiral partition sum

$$Z^{s/r}(q) = \sum_{Q=-(r-s)}^{(s-1)} \sum_{M,N\geq 0} \frac{q^{E(Q;M,N)}}{(q)_M (q)_N} \ . \tag{3.37}$$

The equality of the expressions 3.34 and 3.37 is a new identity of the Rogers-Ramanujan type (see [50, 12, 13] for some similar identities).

3.3 Equilibrium Quantities

3.3.1 Specific Heat

The specific heat of a conformal field theory is well-known to be proportional to the central charge c_{CFT}

$$\frac{C(T)}{L} = \gamma \rho_0 k_B^2 T \ , \quad \gamma = \frac{\pi^2}{6}\, c_{CFT} \ , \tag{3.38}$$

where $\rho_0 = (\hbar v_F)^{-1}$ is the density of states per unit length.
In [39] it was shown that the specific heat for g-on excitations, with energies in the full range $-\infty < \epsilon < \infty$, is in agreement with the central charge $c_{CFT} = 1$ of the corresponding CFT. The same result should of course come out in a picture where we select positive energy electrons and positive energy quasi-holes as our fundamental excitations. In this picture the total energy carried by the edge quasi-particles takes the form

$$E = \rho_0 \int_0^\infty d\epsilon\, \epsilon\, \bar{n}_g(\epsilon) + \rho_0 \int_0^\infty d\epsilon\, \epsilon\, \bar{n}_{\tilde{g}}(\epsilon) \ . \tag{3.39}$$

and the corresponding result for the specific heat is

$$\frac{C(T)}{L} = (\gamma_{g,+} + \gamma_{\tilde{g},+}) \rho_0 k_B^2 T \ , \tag{3.40}$$

where

$$\gamma_{g,+} = \partial_\beta \int_0^\infty d\epsilon\, \epsilon\, \bar{n}_g(\epsilon) \ , \qquad \gamma_{\tilde{g},+} = \partial_\beta \int_0^\infty d\epsilon\, \epsilon\, \bar{n}_{\tilde{g}}(\epsilon) \ . \tag{3.41}$$

It takes an elementary application of the duality relation 3.26 to show that $\gamma_{\tilde{g},+} = \gamma_{g,-}$ and hence

$$(\gamma_{g,+} + \gamma_{\tilde{g},+}) = \gamma_g = \frac{\pi^2}{6} \ , \tag{3.42}$$

confirming once again the value $c_{CFT} = 1$.
We would like to stress that the individual contributions $\gamma_{g,+}$ do depend on g and that only for $g = 1$ (Majorana fermions) $\gamma_{g,+}$ and $\gamma_{g,-}$ are equal. An exact result is [60]

$$\gamma_{g,+} = \frac{\pi^2}{6}\frac{L(\xi^g)}{L(1)} \ , \tag{3.43}$$

with ξ a solution of the algebraic equation

$$\xi^g = 1 - \xi \tag{3.44}$$

and $L(z)$ the Rogers dilogarithm. This gives

$$\gamma_{\frac{1}{2},+} = \frac{\pi^2}{6}\frac{3}{5}, \quad \gamma_{2,+} = \frac{\pi^2}{6}\frac{2}{5}, \quad \gamma_{\frac{1}{3},+} = \frac{\pi^2}{6}\,0.655\ldots, \quad \gamma_{3,+} = \frac{\pi^2}{6}\,0.344\ldots, \quad \text{etc.} \tag{3.45}$$

3.3.2 Hall Conductance

While the specific heat coefficient γ is not sensitive to g, the edge capacitance or, equivalently, the Hall conductance, obviously does depend on the filling fraction ν and thereby on g. In the quasi-particle formulation, this result comes out in a particularly elegant and simple manner.
Let us focus on a $\nu = \frac{1}{p}$ edge and take as our fundamental quasi-particles the edge electron of charge $q = -e$ and statistics $g = p$ and the edge quasi-hole of charge $q = \frac{e}{p}$ and statistics $\tilde{g} = \frac{1}{p}$, all quasi-particles having positive energies only.
Let us first consider zero temperature, where the Haldane distribution functions are step-functions with maximal value $\bar{n}_g = \frac{1}{g}$. If we now put a voltage $V > 0$ the $q < 0$ quasi-particles will see their Fermi energy shift by the amount qV and all available states at energy up to $-qV$ will be filled. The total excess charge on the edge $\Delta Q(V, T = 0)$ that is carried by these excitations equals

$$\Delta Q(V, T = 0) = \frac{1}{g} \cdot q \cdot \rho_0 \cdot (-qV) \tag{3.46}$$

where the factor $\frac{1}{g}$ originates from the maximum of the distribution function and thus represents the statistics properties of the quasi-particles. Clearly, positive-q quasi-particles do not contribute to the response at $T = 0$, $V > 0$.
For the $\nu = \frac{1}{p}$ fqHe edges, the result for $V > 0$ is

$$\Delta Q(V > 0, T = 0) = \frac{1}{p} \cdot (-e) \cdot \rho_0 \cdot (eV) = -\frac{e^2}{p}\rho_0 V \tag{3.47}$$

while for $V < 0$

$$\Delta Q(V < 0, T = 0) = m \cdot \frac{e}{p} \cdot \rho_0 \cdot (-\frac{e}{p}V) = -\frac{e^2}{p}\rho_0 V \ . \tag{3.48}$$

Clearly, the edge capacitance

$$\frac{\Delta Q(V, T=0)}{V} = -\rho_0 \frac{e^2}{p} \tag{3.49}$$

is independent of the sign of V and we establish the correct value of the Hall conductance

$$G = \frac{1}{\rho_0 h} \frac{|\Delta Q|}{V} = \frac{1}{p} \frac{e^2}{h}. \tag{3.50}$$

To show that the results (3.49), (3.50) hold for finite temperatures as well we write the general expression

$$\Delta Q(V,T) = -e\rho_0 \int_0^\infty d\epsilon\, \bar{n}_m(\epsilon + eV) + \frac{e}{m}\rho_0 \int_0^\infty d\epsilon\, \bar{n}_{\frac{1}{p}}(\epsilon - \frac{e}{m}V) \tag{3.51}$$

and evaluate $\partial_\beta \Delta Q(V,T)$. Using once again the duality relations 3.26, we derive

$$\partial_\beta \Delta Q = \frac{e}{p}\,\rho_0 \partial_\beta \int_{-\infty}^\infty d\epsilon\, \bar{n}_{\frac{1}{p}}(\epsilon) \propto \int_0^\infty dx\, \ln(x) \partial_x \bar{n}_{\frac{1}{p}}(x) \tag{3.52}$$

with $x = e^{-\beta\epsilon}$. Using eq. 3.9 the last line turns into

$$\propto \lim_{x_0 \to \infty} \Big[\ln \lambda^+(x) - \bar{n}_{\frac{1}{p}}(x) \ln(x) \Big]_0^{x_0} \tag{3.53}$$

and by using the asymptotic behavior for $x \to \infty$

$$\lambda^+(x) \approx x^p\ , \qquad \bar{n}_{\frac{1}{p}} \approx p \tag{3.54}$$

we conclude that $\partial_\beta \Delta Q$ is indeed zero.

4

Jack Polynomial Technology

The quasi-particle basis (pre-fqH) that we specified in eq. 3.27 has some arbitrariness to it. For example, we could have chosen to act first with the J_{-t} and then with ϕ_{-s}, which would have led to a different set of states. Also, one quickly finds that the pre-fqH states as they stand are not mutually orthogonal. For the purpose of establishing the thermodynamics of the fqH-edge theory, what matters is the counting of the number of states with given charge and energy, and this information can be extracted from the pre-fqH basis. However, for the analysis of more detailed questions, in particular those concerning non-equilibrium transport, the precise form of the multi-quasi-particle states is of crucial importance.

In this chapter we present an 'improved' set of multi-particle states, which are mutually orthogonal and which are faithful to the statistics properties of the quasi-particles ϕ_{-s} and J_{-t}. These fqH-states are obtained by orthogonalizing the states in the pre-fqH basis 3.27. The Calogero-Sutherland Hamiltonian H_{CS} acts lower triangular on the pre-fqH states and therefore the labeling of the pre-fqH states can be used to label the H_{CS} eigenstates. Thus the fqH states or H_{CS} eigenstates are $|\{m_i\}\{n_j\}\rangle^Q$ where Q labels the sector, and $\{m_i\}$ and $\{n_j\}$ are the mode-indices labeling the J's and ϕ's, respectively. From now on we will call the pre-fqH state used to label the fqH-state the head state.

We also consider an alternative basis, the CS-basis, which is easier to handle mathematically. This basis was first proposed by Iso [43], who showed that normal ordered products of vertex operators acting on a vacuum state can be rewritten as a linear combination of Jack-states $|\tilde{q}, \{\mu\}_J\rangle$ or $|\tilde{q}, \{\nu\}_\phi\rangle$. These states are created by linear combinations of products of J_{-t} and ϕ_{-s}-operators, respectively. Alternatively, the same states are obtained when Jack operators are applied to a charged vacuum. Where the Jack operators are Jack polynomials in which the power sums are replaced with modes of the density operator $p_k \rightarrow \rho_k$. The eigenstates of H_{CS} created by these Jack operators have a different charge label $|q, \{\mu\}_J\rangle$ because the Jack operators $J^\beta_{\{\mu\}}$ create neutral excitations over the vacuum $|q, 0\rangle$. In other words q is simply the total charge of a state and $\{\mu\}$ labels a Jack-operator creating bosonic density fluctuation excitations over the charge q ground state. The states in a CS-basis created from the vacuum $|\tilde{q} = q - Mp, 0\rangle$ with a linear combination of products of M

J's are written $|\tilde{q}, \{\mu\}_J\rangle$. The tableau μ contains the same information as the mode-indices of the head state coming from the pre-CS basis.
The pre-CS basis is similar to the pre-fqH basis but instead of using electrons and quasi-holes and a small set of sectors it uses only one type of charged quasi-particles and an infinite set of charged vacua. We will give a more detailed discussion of both bases, we will first explain Jack polynomial technology when we give a short review of the most relevant properties of Jack polynomials. Secondly we use the lower triangular action of H_{CS} on the pre-fqH basis and pre-CS basis to derive the orthogonality of the fqH-basis and the CS-basis. After that we use the lower triangular action of H_{CS} and a duality property of Jack-polynomials to set up a one-to-one correspondence between the fqH-basis and the CS-basis. Then we give vacuum, one-particle, and two particle form factors. The two particle form factors act as a bridge to the last section in which we use the full Jack polynomial technology to obtain selection rules in the CS-basis.

4.1 Jack Polynomials and Jack Operators

In this section we give a quick overview of the most important definitions and theorems on Jack-polynomials that can be found in the literature [76, 3] and show how they can be used to rewrite vertex operators. For those acquainted with the subject let us briefly mention what our conventions will be. We will use the notation of Iso [43] except for the Jack polynomials themselves for which we will introduce two notations to make a clear distinction between the coordinate depend Jack polynomials $P^p_{\{\mu\}}(\{z_i\})$ and the bosonic mode Jack operators $J^{\frac{1}{p}}_{\{\mu\}}\left(\sqrt{p}a_{-k}\right)$, where the a_k are bosonic modes satisfying the Heisenberg algebra. If comparison is made with statements in reference [76, 3] it should be remembered that Stanley and MacDonald use a different labelling of the inner-product $\beta \to \frac{1}{\beta}$ and that Stanley uses a different normalization $v_{\lambda 1^{|\lambda|}} = |\lambda|!$ in [76]. The reason to use Iso's conventions is that the work presented in this chapter is closely related to his work on the Calogero-Sutherland model. Because of this connection the work presented in this chapter which was originally done with the intent of applying it to the fractional quantum Hall effect has important implications for the Calogero-Sutherland model aswell.

4.1.1 Partitions

Every Jack polynomial is labelled with a tableau or partition,

$$\{\lambda\} = \{(i,j) : 1 \leq i \leq l(\lambda), 0 \leq j \leq \lambda_i\} \tag{4.1}$$

with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{L(\{\lambda\})}$ and $L(\{\lambda\})$ denotes the number of entries λ_i in this partition. The length $l(\{\nu\}) \leq L(\{\nu\})$ is the number of non-zero entries or parts λ_i. The reason to introduce two lengths is that Jack polynomials are insensitive to the number of zero-valued parts were in specifying the eigenstates of the H_{CS} the total charge depends on the number of zeros in the partition.

The weight of the tableau is $|\lambda| = \sum_i \lambda_i$. The conjugate partition λ' is obtained by interchanging rows and columns in the tableau representing the partition λ, see figure: 4.1.

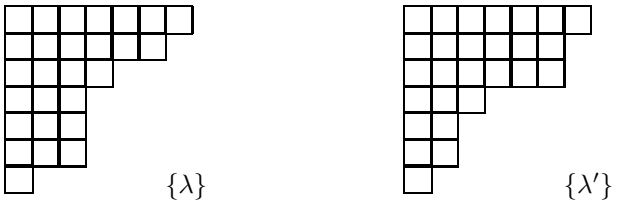

Figure 4.1: Tableaux for the partition $\{\lambda\} = (7, 6, 4, 3, 3, 3, 1)$ and its conjugate $\{\lambda'\} = (7, 6, 6, 3, 2, 2, 1)$.

Another notation for the partition $\{\lambda\} = (0^{l_0}, 1^{l_1}, 2^{l_2}, 3^{l_3}, ...)$ where l_j denotes the number of entries which satisfy $\lambda_i = j$. This notation is very suitable to introduce the quantity z_λ,

$$z_\lambda = \prod_{i \geq 1} i^{l_i} l_i!, \tag{4.2}$$

and we have $L(\{\lambda\}) = l(\{\lambda\}) + l_0 = \sum_{i \geq 0} l_i$.
On the partitions several partial orderings can be defined. For our purposes only the natural or dominance ordering $\geq$ is important, which is defined for partitions of equal weight $|\lambda| = |\mu|$,

$$\{\lambda\} \geq \{\mu\} \Leftrightarrow \text{for all } \; i \geq 1 \; \text{ we have : } \; \sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j. \tag{4.3}$$

4.1.2 Jack Polynomials

There are many well studied complete bases for the ring of symmetric functions, we will need: the monomial symmetric functions, power sums, and Jack-polynomials. The Jack-polynomials will depend on a parameter associated with the inner product defined for the power sums.
Monomial symmetric functions $m_{\{\lambda\}}$ are given by,

$$m_{\{\lambda\}}(\{x_i\}) = \sum_\sigma \prod_i x_i^{\lambda_{\sigma(i)}}, \tag{4.4}$$

where $\sum_\sigma$ denotes the sum over all permutations of the indices i and $\{x_i\}$ is a potentially infinite set of variables.
Power sums are defined by

$$p_{\{\lambda\}}(\{x_i\}) = \prod_{j=1}^{l(\lambda)} p_{\lambda_j}(\{x_i\}) \;\; \text{with} \;\; p_{\lambda_j}(\{x_i\}) = \sum_i x_i^{\lambda_j}, \tag{4.5}$$

and again $\{x_i\}$ is a potentially infinite set of variables.
It is possible to introduce an inner product on the ring of symmetric polynomials,

$$\langle p_{\{\lambda\}}|p_{\{\mu\}}\rangle_\beta = \delta_{\{\lambda\},\{\mu\}}\beta^{-l(\lambda)}z_\lambda \tag{4.6}$$

where β is a rational number. In this inner product we can recognize the inner product on states created by the mode-operators $\rho_k = \sqrt{p}a_k$ if we make the identifications

$$\begin{aligned} |p_k\rangle_\beta &= \tfrac{1}{\sqrt{\beta}}a_{-k}|0\rangle \\ \text{with } \beta &= \nu = \tfrac{1}{p}, \end{aligned} \tag{4.7}$$

where ν denotes the filling fraction. Because of this identification we will label the Jack-polynomials with $\beta = p^{\pm 1}$ instead of using the standard mathematical convention α or Iso's convention β used so far. We will still use β if we make statements in which p and $\frac{1}{p}$ can be interchanged.
The coordinate Jack-polynomials $P^{p^{\pm 1}}_{\{\lambda\}}(\{z_i\})$ are symmetric functions in the coordinates $\{z_i\}$ labeled by partitions λ and a rational number $\beta = p^{\pm 1}$ specifying the inner product 4.6 used on the power sums. They are defined by the following properties:

orthogonality:

$$\langle P^\beta_{\{\lambda\}}(\{z_i\})\,|P^\beta_{\{\nu\}}(\{z_i\})\rangle_\beta = \delta_{\{\lambda\},\{\nu\}}j^\beta_{\{\nu\}}$$

triangularity:

$$P^\beta_{\{\lambda\}}(\{z_i\}) = \textstyle\sum_{\{\mu\}} v_{\lambda,\mu}(\beta)m_{\{\mu\}} \text{ where } v_{\lambda,\mu}(\beta) = 0 \text{ unless } \{\mu\} \le \{\lambda\}$$

normalization:

$$\text{The coefficient } v_{\lambda,\lambda} = 1$$

The undefined function $j^\beta_{\{\nu\}}$ in the definition of the inner product can be shown to satisfy [3].

$$j^\beta_{\{\nu\}} \quad = \quad \prod_{(i,j)\in\{\nu\}} \frac{\beta(\nu'_j - i) + \nu_i - j + 1}{\beta(\nu'_j - i + 1) + \nu_i - j}. \tag{4.8}$$

Jack polynomials satisfy the following duality between $\beta = p$ and $\beta = \frac{1}{p}$:

$$P^p_{\{\lambda'\}}\left(\left\{\frac{p_n}{p}\right\}\right) = (-1)^{|\lambda|}j^p_{\{\lambda\}}P^{\frac{1}{p}}_{\{\lambda\}}(-\{p_n\}), \tag{4.9}$$

where we have written $P^\beta_{\{\mu\}}(\{z_i\})$ as a function of the power sums which is possible because the power sums are a basis for the symmetric functions.

From this duality property or the explicit form of the inner products $j^{\beta=p^{\pm 1}}_{\{\nu\}}$ follows

$$j^{p}_{\{\nu\}} j^{\frac{1}{p}}_{\{\nu'\}} = 1, \tag{4.10}$$

which is important for the explicit calculation of inner products on fqH-states as will become clear later.
An elementary property of the Jack-polynomials is

$$\prod_{i,j}(1 - x_i y_j) = \sum_{\{\lambda\}} (-1)^{|\lambda|} P^{\beta}_{\{\lambda\}}\left(\{x_i\}\right) P^{\frac{1}{\beta}}_{\{\lambda'\}}\left(\{y_j\}\right), \tag{4.11}$$

this property is crucial for our purposes because it will enable us to rewrite the vertex-operators describing the charged excitations in the $U(1)$-CFT for the fqH-edge to Jack-states.
For integer β an alternative inner product [3] on the Jack-polynomials $P^{\beta}_{\{\mu\}}$ depending on only a finite set of coordinates $\{z_i\} = \{z_1, ..., z_n\}$ is given by

$$\langle\langle P^{\beta}_{\{\nu\}}\left(\{z_i\}\right) | P^{\beta}_{\{\mu\}}\left(\{z_i\}\right)\rangle\rangle =$$

$$\left(\prod_{i=1}^{n} \oint \frac{dz_i}{2\pi i}\right) \Delta^{\beta}(\{z_i^{-1}\})\Delta^{\beta}(\{z_i\}) P^{\beta}_{\{\nu\}}\left(\{z_i^{-1}\}\right) P^{\beta}_{\{\mu\}}\left(\{z_i\}\right) \tag{4.12}$$

where $\Delta^{\beta}(\{x_i\}) = \prod_{i<j}(x_i - x_j)^{\beta}$ denotes a generalized Vandermonde determinant. Although it is also possible to define this innerproduct for fractional β [3], we will use it in this form for integer β. This inner product satisfies orthogonality and can be written $p^{\beta}_{\{\nu\},n}\delta_{\mu\nu}$ where the function $p^{\beta}_{\{\nu\},n}$ is given by

$$p^{\beta}_{\{\nu\},n} = \prod_{1\le i<j\le n} \frac{\Gamma(\xi_i - \xi_j + \frac{1}{\beta})\Gamma(\xi_i - \xi_j - \frac{1}{\beta} + 1)}{\Gamma(\xi_i - \xi_j)\Gamma(\xi_i - \xi_j + 1)}$$

$$\text{with } \xi_i = \lambda_i + \frac{1}{\beta}(n - i) \text{ for } 1 \le i \le n. \tag{4.13}$$

The function $p^{\beta}_{\{\nu\},n}$ will drop out of our final results so we mention it here only for completeness.
In the above we introduced the most basic properties of Jack-polynomials. Other properties will be given when needed.

4.1.3 Vertex Operators

We now have the tools available to rewrite the vertex operators in terms of Jack-polynomials and Jack-operators. We still have to introduce the latter but one idea behind it we presented in eq. 4.7 when we pointed at the analogy between the inner product on power sums which we used in the definition of the Jack-polynomials and the inner product on the modes of the density operator which we introduced in the last section of chapter 1.

The second idea needed is to rewrite the right hand side of eq. 4.11 in to a form that shows close analogy with a vertex operator,

$$\prod_{ij}(1-x_iz_j)=\exp\left(\ln\prod_{ij}(1-x_iz_j)\right)=\exp\left(-\sum_{n=1}^{\infty}\sum_j\frac{p_n(\{x_i\})z_j^n}{n}\right). \quad (4.14)$$

If we replace p_n by $\frac{a_{-n}}{\sqrt{p}}$ we recognize a part of a normal ordered product of the vertex operators $\phi(z_1)....\phi(z_N)$, which would be

$$\begin{aligned}:\phi(z_1)....\phi(z_N):=&\\ &\exp\left(-\frac{1}{\sqrt{p}}\sum_{n=1}^{\infty}\sum_{j=1}^{N}\frac{a_{-n}z_j^n}{n}\right)\exp\left(i\frac{N}{\sqrt{p}}\varphi_0\right)\\ &\times\left(\prod_{j=1}^{N}z_j^{-\frac{1}{\sqrt{p}}p_0}\right)\exp\left(\frac{1}{\sqrt{p}}\sum_{n=1}^{\infty}\sum_{j=1}^{N}\frac{a_nz_j^{-n}}{n}\right),\end{aligned} \quad (4.15)$$

where in addition to the modes found in chapter 1 we introduced the zero-modes φ_0, p_0 which are needed to describe the charge structure and which satisfy $[\varphi_0, p_0] = i$. If we let this operator act on a highest weight states or q vacuum $|q,0\rangle$ all the a_n with positive mode index will annihilate the vacuum, p_0 will measure the charge of the vacuum, and $\exp\left(iN\frac{q'}{\sqrt{p}}\varphi_0\right)$ will change the charge of the vacuum by Nq' where the charge is given in units $\frac{1}{p}$ and we get

$$\begin{aligned}:\phi(z_1)...\phi(z_N):|q,0\rangle=&\\ &\exp\left(-\frac{1}{\sqrt{p}}\sum_{n=1}^{\infty}\sum_{j=1}^{N}\frac{a_{-n}z_j^n}{n}\right)\prod_{j=1}^{N}z_j^{\frac{q}{p}}|q+N,\rangle.\end{aligned} \quad (4.16)$$

Now we can use eq. 4.14 and eq. 4.11 to rewrite the r.h.s. to

$$\sum_{\{\lambda\}}(-1)^{|\lambda|}J^p_{\{\lambda\}}\left(\left\{\frac{a_{-n}}{\sqrt{p}}\right\}\right)P^{\frac{1}{p}}_{\{\lambda'\}}(\{z_j\})\prod_{j=1}^{N}z_j^{\frac{q}{p}}|q+N\rangle, \quad (4.17)$$

where we used $J^p_{\{\lambda\}}\left(\left\{\frac{a_{-n}}{\sqrt{p}}\right\}\right)$ to denote a Jack-polynomial in which the power-sums are replaced by bosonic modes. And using the same reasoning as before it can be found that

$$\begin{aligned}:J(z_1)...J(z_N):|q,0\rangle=&\\ &\sum_{\{\lambda\}}(-1)^{|\lambda|}J^{\frac{1}{p}}_{\{\lambda\}}(\{\sqrt{p}a_{-n}\})P^p_{\{\lambda'\}}(\{z_j\})\prod_{j=1}^{N}z_j^q|q-Np\rangle.\end{aligned} \quad (4.18)$$

From now on we will drop the explicit reference to the bosonic modes in the Jack-operators and we will simply write,

$$J^{\frac{1}{p}}_{\{\lambda\}} = J^{\frac{1}{p}}_{\{\lambda\}}\left(\{\sqrt{p}a_{-n}\}\right)$$
$$J^{p}_{\{\lambda\}} = J^{p}_{\{\lambda\}}\left(\{\frac{1}{\sqrt{p}}a_{-n}\}\right). \tag{4.19}$$

It can be shown [43] that every Jack-operator creates an eigenstate of the the Calogero-Sutherland Hamiltonian. In fact the Calogero-Sutherland Hamiltonian cast in the language of CFT as we will use in the next section was obtained by requiring it to reproduce the eigenvalues of the coordinate Calogero-Sutherland model when acting on a polynomial in which the power sums are replaced by density modes.

4.2 H_{CS} and the Improved Set of States

To obtain an orthonormal basis from the pre-fqH basis 3.27 we use the Calogero-Sutherland Hamiltonian H_{CS} as an auxiliary operator for which the eigenstates and the eigenvalues are known and which acts lower triangular on the pre-fqH basis. Because of the lower triangular action of H_{CS} on the pre-fqH basis we can use the states in it to label the eigenstates.
To specify the operator H_{CS}, we employ the density operator $\rho(z) = \sum_l \rho_l z^{-l-1}$, which already featured in our formula 1.15. Following [43], we define

$$H_{CS} = \frac{p-1}{p}\sum_{l=0}^{\infty}(l+1)(ip\rho)_{-l-1}(ip\rho)_{l+1} + \frac{1}{3p}\left[(ip\rho)^3\right]_0 \tag{4.20}$$

and where the second term on the r.h.s. denotes the zero-mode of the normal ordered product of three factors $(ip\rho)(z)$. As a first result, one finds the following action of H_{CS} on vertex-operators creating a single quasi-particle of charge $\frac{e}{p}$ or $-e$

$$\begin{aligned}
[H_{CS}, \phi_{-n}] &= h_\phi(n)\,\phi_{-n} + \frac{p-1}{p}\sum_{l\geq 0} 2(l+1)\phi_{-n-l-1}(\partial\rho)_{l+1} \\
&\quad \text{with} \qquad h_\phi(n) = \left[\frac{1}{3p} + pn(n+\frac{1}{p})\right] \\
&\quad \text{and} \\
[H_{CS}, J_{-m}] &= h_J(m)J_{-m} - (p-1)\sum_{l\geq 0} 2(l+1)J_{-m-l-1}(\partial\rho)_{l+1} \\
&\quad \text{with} \qquad h_J(m) = \left[-\frac{p^2}{3} - m(m+p)\right] .
\end{aligned} \tag{4.21}$$

Both $\phi_{-n}|q,0\rangle$ and $J_{-m}|q,0\rangle$ diagonalize H_{CS}. We would like to stress that this is non-trivial. If one evaluates H_{CS} on any vertex operator $\phi^{(q)}$ (of charge $q\frac{e}{p}$), one typically runs into the field product $(T\phi^{(q)})(z)$, where $T(z) = -\frac{p}{2}(\rho)^2(z)$ is the stress-energy of the scalar field $\varphi = \sqrt{p}\rho$. Only for $q = 1$ and $q = -p$ do such terms cancel and do we find that the quasi-particle states are eigenstates of H_{CS}.

We can now construct eigenstates of H_{CS} which contain several ϕ or J-quanta. What one then finds is that the simple product states such as the pre-fqH states 3.27 are not H_{CS} eigenstates, but that they rather act as head states that need to be supplemented by a tail of sub-leading terms. As an example, one finds two-ϕ eigenstates to be of the form [12]

$$|(n_2,n_1)\rangle = \phi_{-\frac{3}{2p}-n_2}\phi_{-\frac{1}{2p}-n_1}|0\rangle + \sum_{l=1}^{\infty} a_l \left[\phi_{-\frac{3}{2p}-n_2-l}\phi_{-\frac{1}{2p}-n_1+l}|0\rangle\right] \qquad (4.22)$$

with coefficients a_l that can be computed. The connection of the coefficients a_l with the Jack polynomials that feature in the eigenfunctions in CS quantum mechanics has been made explicit in [12]. For the H_{CS} eigenstate headed by the multi-particle state from the pre-fqH basis 3.27 (with unit coefficient), we shall use the notation

$$|\{m_j\},\{n_i\}\rangle^Q \qquad (4.23)$$

which is the H_{CS} eigenstate corresponding to the eigenvalue

$$h(\{m_j\},\{n_i\},Q) = \left[\frac{Q^3}{3p} + \sum_{j=1}^{M} h_J\left((2j-1)p+m_j\right) + \sum_{i=1}^{N} h_\phi\left(\frac{1}{p}(2i-1)+n_i\right)\right]. \qquad (4.24)$$

The states 4.23, with the m_j, n_i and Q as specified in eq. 3.27, form a complete and orthogonal basis for the chiral Hilbert space.

An important point to notice here is that the dependence of the eigenvalues of H_{CS} on the head states involves no mixed products of mode-indices. Furthermore, the eigenvalues depend directly on the full mode-indices of the head state and only through the full mode-indices on the shifts $\{m_i\},\{n_i\}$ with respect to the state with the lowest energy. Whether a mode-operator is an allowed mode-operator will depend on the vacuum. However, if a mode operator, say J_{-m}, appears in the head state it will signal the presence of a term $h_J(m)$ in the eigenvalue regardless of the vacuum charge. The only dependence of the eigenvalue on the vacuum charge is through a term that is independent of the mode-indices. These properties will be crucial for constructing the mapping between the fqH-basis and the CS-basis which we introduce in the next section.

4.3 Correspondence Between the fqH-basis and the CS-basis

We only briefly mentioned the CS-bases, but after the discussion of the fqH-basis and the introduction of the Jack-operators we can easily introduce the CS-bases and set-up a one to one mapping. In [43] it was shown that the Jack-operators create eigenstates of H_{CS}. The eigenvalue of the the state

$$|q, \{\mu\}_J\rangle = J^{\frac{1}{p}}_{\{\mu'\}}|q\rangle \tag{4.25}$$

correspond to the eigenvalue of the state headed by the pre-CS state

$$J_{-m_{\tilde{M}}+q+\frac{p}{2}} J_{-m_{\tilde{M}-1}+q+\frac{3p}{2}} \dots J_{-m_1+q+\frac{2\tilde{M}p-p}{2}} |q+\tilde{M}p\rangle \tag{4.26}$$

with $m_j = \mu_{\tilde{M}-j+1} \geq 1$ and $\tilde{M} = l(\{\mu\})$. The reason to choose the $m_j \geq 1$ is that mode operators with $m_j = 0$ take us from one vacuum to the other. The states created by these mode operators are therefore already accounted for by allowing many different vacua.
And similarly we have the correspondence between

$$|q, \{\nu\}_\phi\rangle = J^p_{\{\nu'\}}|q\rangle \tag{4.27}$$

and the state headed by the pre-CS state

$$\phi_{-n_{\tilde{N}}-q+\frac{1}{2p}} \phi_{-n_{\tilde{N}-1}-q+\frac{3}{2p}} \dots \phi_{-n_1-q+\frac{2\tilde{N}-1}{2p}} |q-\tilde{N}\rangle. \tag{4.28}$$

with $n_i = \nu_{\tilde{N}-i+1} \geq 1$ and $\tilde{N} = l(\{\nu\})$. For the same reason as before, choose the n_i to satisfy $n_i \geq 1$.
Let us first rewrite fqH-states which contain only one type of mode-operators to the CS-bases,

$$\begin{aligned} |\{\}, \{n_i\}\rangle^Q &= |Q+N, \{\nu\}_\phi\rangle \\ |\{m_j\}, \{\}\rangle^Q &= |Q-pM, \{\mu\}_J\rangle. \end{aligned} \tag{4.29}$$

with $\nu_i = n_{N-i+1} \geq 1$ for $\tilde{N} - i \geq 0$ and similarly $\mu_j = m_{M-j+1} \geq 1$ for $\tilde{M} - j \geq 0$.
The duality 4.9 on the Jack polynomials leads to the following duality relation for the Jack operators,

$$J^p_{\{\lambda'\}} = (-1)^{|\lambda|} j^p_{\{\lambda\}} J^{\frac{1}{p}}_{\{\lambda\}} \tag{4.30}$$

This relation enables us rewrite the states $|\{\}, \{n_i\}\rangle^Q$ to either the $|Q+N, \{\nu'\}_J\rangle$ or the $|Q+N, \{\nu\}_\phi\rangle$ form.

We can for example rewrite,

$$\begin{aligned} |\{\},\{n_i\}\rangle^Q &= |Q+N,\{\nu\}_\phi\rangle \\ &= (-1)^{|\nu|} j^p_{\{\nu'\}} |Q+N,\{\nu'\}_J\rangle \end{aligned} \tag{4.31}$$

where $\nu_i = n_{\tilde{N}-i+1}$ for $i \leq \tilde{N} = l(\{n_i\})$ and $N = L(\{\nu\})$ and zero otherwise. The last state is equivalent to the state which has

$$J_{-\nu'_{n_1}+Q+N+\frac{p}{2}} J_{-\nu'_{n_1-1}+Q+N+\frac{3p}{2}} ... J_{-\nu'_N+Q+N+\frac{(2n_1-1)p}{2}} |Q+N+n_1 p\rangle \tag{4.32}$$

as its leading state. If we look at the m_j in the state $|\{m_j\},\{n_i\}\rangle^Q$ we see that when the associated mode-operators act on the vacuum $|Q+N\rangle$ mode-indices which are between $Q-\frac{p}{2}$ and $Q-\frac{p}{2}+N$ are allowed as well. When we make the mapping to the CS-basis this will give an increase of the parts appearing in the partition by N i.e. $m_i \to m_i + N$. And the full correspondence, which we illustrated in figure 4.2, is

$$\begin{gathered} |\{m_j\},\{n_i\}\rangle^Q = |Q+N-pM,\{\sigma\}\rangle \\ \text{with}: \ \{\sigma\} = (\{\mu\}+N^M) \cup \{\nu'\}, \end{gathered} \tag{4.33}$$

where $M = L(\{\mu\})$ and the sum of the partitions is $\{\mu\}+N^M = (\mu_1+N, \mu_2+N, .., \mu_M+N)$, and the cup product $\{\lambda\}\cup\{\rho\}$ denotes the partition obtained from sorting the parts $(\lambda_1, ..., \lambda_S, \rho_1, ..., \rho_R)$ in descending order.
The mapping from the CS-basis back to the fqH-basis is slightly more complicated. If we look at the fqH-basis we see that the highest allowed mode-index for a J_s is $s = -\frac{p}{2}$ from which we get the condition that coming from the left the last operator in $|q,\{\sigma\}_J\rangle$ belonging to the fqH-basis is labeled with the last index σ_j which satisfies

$$-\frac{1}{2}p \geq q + pj - \frac{p}{2} - \sigma_j. \tag{4.34}$$

with $S = l(\{\sigma\})$. The sector is determined by looking at the charge of the state which remains if all operators left from and at the position j are removed. With the use of duality this state can rewritten,

$$|q+pj,\{\nu'\}_J\rangle \propto |q+pj,\{\nu\}_\phi\rangle \tag{4.35}$$

with

$$\nu'_i = \sigma_{i+j} \ \text{ for } \ i = 1 \ldots l(\{\sigma\}) - j \tag{4.36}$$

The ϕ-operators in the head state of $|q+pj,\{\nu\}_\phi\rangle$ act on the charge $\tilde{Q} = q+pj-\sigma_{j+1}$ vacuum. If the charge $\tilde{Q}$ of this vacuum is smaller or equal to zero it is equivalent to the charge sector $Q = \tilde{Q}$ of the fqH-state. If, however,

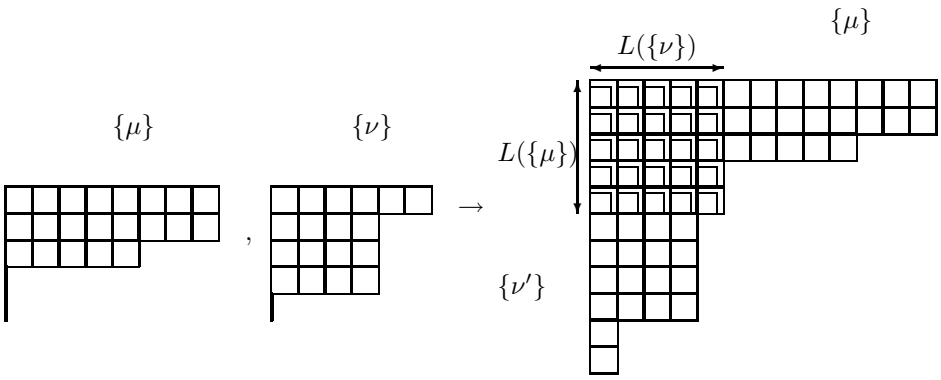

Figure 4.2: Example of the mapping from a state from the fqH-basis $|(8,8,5,0,0),(6,4,4,4,0)\rangle^0$ to a state in the CS-basis $|5-5p,(13,13,10,5,5,4,4,4,4,1,1)_J\rangle$. The tails at the bottom of the fqH-partitions denote the m_j, n_i which are zero.

$\tilde{Q}$ is larger then zero the state was created from the charge zero sector using ϕ-operators with the highest allowed mode index. From this argument we obtain that the state $|q,\{\sigma\}_J\rangle$ from the CS-basis can be rewritten to $|\{m_i\}\{n_i\}\rangle^Q$ with the following rules for selecting the charge sector,

$$\begin{aligned} -p+1 \leq \tilde{Q} \leq 0 \quad &\Rightarrow \quad Q=\tilde{Q} \text{ and } \tilde{N}=N=\sigma_{j+1} \\ \tilde{Q}>0 \quad &\Rightarrow \quad Q=0 \text{ and } N=\tilde{N}+\tilde{Q}, \tilde{N}=\sigma_{j+1}. \end{aligned} \tag{4.37}$$

The mode indices $\{m_i\}$ and $\{n_i\}$ are given by,

$$\begin{gathered} m_i=\sigma_{j-i+1} \text{ for } i=1\ldots j \\ n_i=\nu_{N-i+1} \text{ for } i=1\ldots N, \\ \text{with} \\ \nu'_i=\sigma_{i+j} \text{ for } i=1\ldots l(\{\sigma\})-j \quad . \end{gathered} \tag{4.38}$$

And j labels the smallest σ_i that satisfies

$$-\frac{1}{2}p \geq q+p\frac{2i-1}{2}-\sigma_i. \tag{4.39}$$

4.4 Applications of JPT

4.4.1 Norms for the fqH-basis

Of importance for later calculations are the norms of the states 4.23 and the matrix elements of physical operators between these states. For the explicit evaluation of such quantities we used the connection with Jack polynomials.
As an example, we focus on multi quasi-hole states $|\{n_i\}\rangle$. To make contact with the Jacks, we view the ordered set $\{n_i\}$ as a Young tableau ν. The norm-squared of the state $|\{n_i\}\rangle$ then becomes

$$\langle\{n_i\}|\{n_i\}\rangle = j^p_{\{\nu'\}}. \tag{4.40}$$

Explicit examples are

$$\begin{aligned}
\langle(n_1)|(n_1)\rangle &= j_{(1^{n_1})} = \frac{(n_1+\frac{1}{p}-1)(n_1+\frac{1}{p}-2)\ldots\frac{1}{p}}{n_1(n_1-1)\ldots 1} \\
\langle(n_2,n_1)|(n_2,n_1)\rangle &= j_{(2^{n_1},1^{n_2-n_1})} \\
&= \frac{(n_2+\frac{2}{p}-1)(n_2+\frac{2}{p}-2)\ldots(n_2+\frac{2}{p}-n_1)}{(n_2+\frac{1}{p})(n_2+\frac{1}{p}-1)\ldots(n_2+\frac{1}{p}-n_1+1)} \\
&\quad\times\frac{(n_2-n_1+\frac{1}{p}-1)\ldots\frac{1}{p}}{(n_2-n_1)\ldots 1}\,\frac{(n_1+\frac{1}{p}-1)\ldots\frac{1}{p}}{n_1\ldots 1} \\
&\text{etc.}
\end{aligned} \tag{4.41}$$

In the limit where all $n_i \gg 1$, one finds

$$\langle\{n_i\}|\{n_i\}\rangle \approx \prod_{i=1}^{N} \frac{{n_i}^{\frac{1}{p}-1}}{\Gamma(\frac{1}{p})}\ . \tag{4.42}$$

Similarly the norm-squared of the edge-electron state $|\{m_i\}\rangle$ is

$$\langle\{m_i\}|\{m_i\}\rangle = j^{\frac{1}{p}}_{\{\mu'\}} \tag{4.43}$$

In the limit where all $m_i \gg 1$, one finds

$$\langle\{m_i\}|\{m_i\}\rangle \approx \prod_{i=1}^{M} \frac{{m_i}^{p-1}}{\Gamma(p)}\ . \tag{4.44}$$

And in general the norm will be given by

$$\begin{aligned}
N_{\{m_j;n_i\}} = \big||\{m_j\}\{n_i\}\rangle^Q\big|^2 &= (j^p_{\{\nu'\}})^2 j^{\frac{1}{p}}_{\{\sigma'\}} \\
\text{with}:\ \{\sigma\} &= (\{\mu\}+N^M)\cup\{\nu'\},
\end{aligned} \tag{4.45}$$

and $\{\nu\}$ and $\{\mu\}$ are the tableaus corresponding to $\{n_i\}$ and $\{m_j\}$, respectively. This norm can be factorized into the original norms times an extra factor associated with the added partition N^M

$$N_{\{m_j;n_i\}} = j^p_{\{\nu'\}} j^{\frac{1}{p}}_{\{\mu'\}} \prod_{(i,j)\in N^M} \frac{(\nu'_j - i + M) + p(\mu_i - j + N + 1)}{(\nu'_j - i + 1 + M) + p(\mu_i - j + N)}. \tag{4.46}$$

for N, M, p of order 1 and all the $m_i, n_i \gg 1$ we see that the norm factorizes into the asymptotic norms of the original partitions.

4.4.2 Zero Temperature Form Factors

We start by considering the simplest non-vanishing form factors of the basic electron operators $J(z)$, $J^\dagger(z)$ against the multi-particle states in the fqH basis:

$$\begin{aligned} \langle 0|J_{\frac{p}{2}+m}|(n_p,\ldots,n_2,n_1)\rangle_N &= f_J(n_p,\ldots,n_1)\,\delta_{m,n_p+\ldots+n_1}\ , \\ \langle 0|J^\dagger_{+\frac{p}{2}+m}|(m_1)\rangle_N &= f_{J^\dagger}(m_1)\,\delta_{m,m_1}\ , \end{aligned} \tag{4.47}$$

where the subscript N indicates that the state has been properly normalized. Because $f_{J^\dagger}$ is the inner product between a Jack operators and a normalized Jack operator we immediately find

$$f_{J^\dagger}(m_1) = \left[C^{(-p)}_{m_1}\right]^{\frac{1}{2}}\ , \tag{4.48}$$

where $C^{(-p)}_m$ are the expansion coefficients in $(1-x)^{-p} = \sum_m C^{(-p)}_m x^m$. We briefly explain the exact evaluation of the form factor $f_J(n_p,\ldots,n_1)$ as defined in eq. 4.47. Let us consider the special case $p = 2$ first. In that case the operator $J(z)$ has conformal dimension 1 and may be identified with one of the currents of the affine Kac-Moody algebra $\widehat{su(2)}_1$. By exploiting the OPE

$$\phi(w_1)\phi(w_2) = (w_1 - w_2)^{+\frac{1}{2}}\left[J^\dagger(w_2) + \mathcal{O}(w_1 - w_2)\right] \tag{4.49}$$

one obtains

$$J^\dagger(w_2) = \oint_{C_{w_1}} \frac{dw_2}{2\pi i}(w_1 - w_2)^{-\frac{3}{2}}\phi(w_1)\phi(w_2)\ . \tag{4.50}$$

Using the expansion formula 4.18 we obtain

$$J^\dagger(w_2)|\,0\,\rangle = \sum_{n_2,n_1} P^{\frac{1}{2}}_{(n_2,n_1)}\,(w,w)\,|q=2,(n_2,n_1)_\phi\rangle \tag{4.51}$$

and it follows that

$$\langle 0|J_{1+m}\,|q=2,(n_2,n_1)\rangle_N = \delta_{m,n_2+n_1}\ \left[j_{(2^{n_1},1^{n_2-n_1})}\right]^{\frac{1}{2}}\ P^{\frac{1}{2}}_{(n_2,n_1)}\,(1,1)\ , \tag{4.52}$$

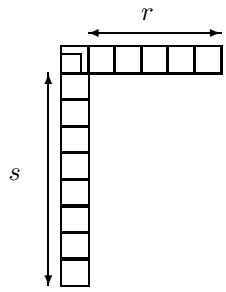

Figure 4.3: A hook $(r|s)$

with j_λ as in eq. 4.8. For general p one obtains a similar result in terms of Jack polynomials with label $\frac{1}{p}$. Using the explicit result [76, 55]

$$P^p_{\{n_i\}}\left(z_1=\ldots=z_p=1\right)=\prod_{i=0}^{p-1}\frac{\Gamma(\frac{1}{p})}{\Gamma(1-\frac{i}{p})}\prod_{i<j}\frac{\Gamma(n_j-n_i+\frac{j-i+1}{p})}{\Gamma(n_j-n_i+\frac{j-i}{p})}\ , \qquad (4.53)$$

together with the result 4.42 for the j_λ, we derive the following asymptotic form for $n_i \gg 1$

$$f_J(n_p,\ldots,n_1)=\frac{[\Gamma(\frac{1}{p})]^{p/2}}{\prod_{i=0}^{p-1}\Gamma(1-\frac{i}{p})}\frac{\prod_{i<j}\left(n_j-n_i\right)^{\frac{1}{p}}}{\prod_{i=1}^{p}n_i^{\frac{(p-1)}{2p}}}\ . \qquad (4.54)$$

4.4.3 Finite Temperature Form Factors

As a first example of the power of combining Jack-polynomials and Jack-operators we will analyze the relation of $J_{-m+Q-\frac{p}{2}}\phi_{-n-\frac{Q}{p}-\frac{1}{2p}}|Q\rangle$ with the Jack-states $|Q-p+1,(r|s)_J\rangle$. Where $(r|s)$ is a short notation for the tableau $(r+1,1^s)$. Because $\phi_{-n-\frac{Q}{p}-\frac{1}{2p}}|Q\rangle$ is equivalent to a Jack-state we immediately obtain information on the form factor

$${}^Q_N\langle(\bar{n}_1)|J^\dagger_{\frac{p}{2}+m-Q}|(m_1;n_1)\rangle^Q_N=\left[\frac{N_{\{m_1;n_1\}}}{N_{\bar{n}_1}}\right]^{\frac{1}{2}}\ f_{\phi|J\phi}(\bar{n}_1,m_1,n_1)\,\delta_{m,m_1+n_1-\bar{n}_1}\ . \qquad (4.55)$$

from this analysis.
Let us start with rewriting the product of a single electron operator and a single quasi-hole operator,

$$J(z)\phi(w)|Q\rangle=$$

$$z^{-Q}w^{\frac{Q}{p}}(z-w)^{-1}\left(\sum_{\{\lambda\}}(-1)^{|\lambda|}P^p_{\{\lambda\}}\left(z\right)J^{\frac{1}{p}}_{\{\lambda'\}}\right)$$

$$\times \quad \left(\sum_{\{\rho\}} (-1)^{|\nu|} P^{\frac{1}{p}}_{\{\rho\}}(w)\, J^{p}_{\{\rho'\}} \right) |Q-p+1\rangle. \tag{4.56}$$

Where we used

$$J(z)\phi(w) = (z-w)^{-1} : J(z)\phi(w) : \tag{4.57}$$

and added factors $z^{-Q}, w^{\frac{Q}{p}}$ which are due to the operators at $z_i = 0$ creating the vacuum $|Q\rangle$ from the zero charge vacuum $|0\rangle$.
Now use duality and the fact that

$$P^{\beta}_{\{\lambda\}}(\{z,0,0,...\}) = \begin{array}{ll} z^{\lambda_1} & \text{for } \lambda_2 = 0 \\ 0 & \text{for } \lambda_2 \neq 0 \end{array} \tag{4.58}$$

to rewrite the right hand side of 4.56 to

$$(z-w)^{-1} z^{-Q} w^{\frac{Q}{p}} \sum_{r,s=0}^{\infty} z^r (-1)^r w^s j^p_{(s-1|0)} J^{\frac{1}{p}}_{(0|r-1)} J^{\frac{1}{p}}_{(s-1|0)} |Q-p+1\rangle. \tag{4.59}$$

If we multiply this expression by $(z-w)z^{-m+Q}w^{-n-\frac{Q}{p}}$ and take contour integrals over contours encircling the origin (with $z > w$) we obtain an expression for

$$J_{Q+2-m-\frac{p}{2}} \phi_{-\frac{2Q+1}{2p}-n+1} |Q\rangle - J_{Q+1-m-\frac{p}{2}} \phi_{-\frac{2Q+1}{2p}-n+2} |Q\rangle. \tag{4.60}$$

This can be cast in a better form with the help of

$$J^{\frac{1}{p}}_{(0|r-1)} J^{\frac{1}{p}}_{(s-1|0)} = \frac{1}{s^{\frac{1}{p}}_{(s-1|0)}} \left(\frac{s^{\frac{1}{p}}_{(s-1|r)} J^{\frac{1}{p}}_{(s-1|r)}}{s^{\frac{1}{p}}_{(s|r-1)}(s+pr)} + r J^{\frac{1}{p}}_{(s|r-1)} \right) \tag{4.61}$$

where

$$s^{\frac{1}{p}}_{\{\lambda\}} = \prod_{(i,j)\in\{\lambda\}} \lambda'_j - i + 1 + p(\lambda_i - j). \tag{4.62}$$

This quantity satisfies

$$s^{\frac{1}{p}}_{(m-1|n-2)} = s^{\frac{1}{p}}_{(m-2|0)} s^{\frac{1}{p}}_{(0|n-2)} \frac{n-1+p(m-2)}{m-1}. \tag{4.63}$$

For the expression 4.60 we obtain

$$\frac{(-1)^{m-1}}{j^p_{(m-2|0)}} \left(\frac{p(n-1)(p(n-2)+m)}{(1+p(n-2)((m-1)+p(n-1))} J^p_{(n-2|m-1)} + J^p_{(n-1|m-2)} \right). \tag{4.64}$$

And from this we obtain the expansion formula

$$: J(z)\phi(w) : |Q\rangle = \sum_{(m_1,n_1)} r_{m_1,n_1}(z,w)|(m_1;n_1)\rangle^Q \tag{4.65}$$

with

$$r_{m,n}(z,w) = z^{m+1}w^n - \frac{m+1+pn}{m+p(n+1)} z^m w^{n+1}. \tag{4.66}$$

This expansion formula leads to the following result for the form factor.

$$f_{\phi|J\phi}(\bar{n}_1, m_1, n_1) = \left[\delta_{\bar{n}_1=n_1} - \frac{p-1}{m_1+p(n_1+1)}\Theta(\bar{n}_1 > n_1)\right] . \tag{4.67}$$

For the $p = 2$ we used the mode-operator algebra to confirm this result.

4.4.4 More Finite Temperature Form Factors

We now consider more general form factors for the edge electron 'annihilation operator' $J^\dagger(z)$. The simplest form factor with a 2-particle in-state is

$$\begin{aligned} &{}^Q_N\langle(\bar{m}_1)|J^\dagger_{\frac{3p}{2}+m}|(m_2,m_1)\rangle^Q_N = \\ &\left[\frac{N_{(m_2,m_1)}}{N_{(\bar{m}_1)}}\right]^{\frac{1}{2}} f_{J|JJ}(\bar{m}_1, m_2, m_1)\, \delta_{m,m_2+m_1-\bar{m}_1} . \end{aligned} \tag{4.68}$$

Using the expansion formula 4.18 and the relation

$$J(z)J(w) = (z-w)^p : J(z)J(w) : \tag{4.69}$$

we derive

$$\begin{aligned} &f_{J|JJ}(\bar{m}_1, m_2, m_1) = \\ &\left[\frac{1}{N_{(m_2,m_1)}}\right]^{\frac{1}{2}} \sum_{l=0}^{p} C_l^{(p)}\, {}_N\langle(m_2,m_1), -2p| - 2p, (m_2+m_1-\bar{m}_1+l)(\bar{m}_1-l)\rangle, \end{aligned} \tag{4.70}$$

where

$$| - 2p, (m_2+m_1-\bar{m}_1+l)(\bar{m}_1-l)\rangle = J^{\frac{1}{p}}_{(m_2+m_1-\bar{m}_1+l)'} J^{\frac{1}{p}}_{(\bar{m}_1-l)'}| - 2p\rangle. \tag{4.71}$$

The duality and the triangularity properties of Jack polynomials $v_{\lambda\mu} \neq 0$ only for $\mu \leq \lambda$ give us the restriction

$$f_{J|JJ}(\bar{m}_1, m_2, m_1) \neq 0 \Leftrightarrow m_1 - p \leq \bar{m}_1 \leq m_2. \tag{4.72}$$

Here we only indicated the steps needed to obtain this result, they are analogous to the steps made in deriving $f_{\phi|J\phi}$ and we will give a full account of these steps

in the next section where we derive general selection rules. It is also possible to see the results in this section as a special case of equation 4.88 in the next section.
Specializing to $p = 1, 2$, we have the following explicit results

$$p = 1: \qquad f_{J|JJ}(\bar{m}_1, m_2, m_1) = [\delta_{\bar{m}_1 = m_2} - \delta_{\bar{m}_1 = m_1 - 1}]$$

$$p = 2: \qquad f_{J|JJ}(\bar{m}_1, m_2, m_1) =$$

$$\left[\delta_{\bar{m}_1 = m_2} + \delta_{\bar{m}_1 = m_1 - 2} - \frac{2}{m_2 - m_1 + 1}\Theta(m_1 - 2 < \bar{m}_1 < m_2)\right] . \quad (4.73)$$

While the results for $p = 1$ are a direct consequence of the Wick theorem, the expressions 4.67 and 4.73 for $p \neq 1$ show that the 'ideal gas interpretation' is no longer applicable for general p: both form factors can be non-vanishing for a process where the energy of the electron annihilation operator $J^\dagger$ does not match the incoming electron energies m_2 or m_1. The energy difference is compensated by a second 'spectator particle'. Furthermore, the form factors $f_{J|JJ}$ and $f_{\phi|J\phi}$ are not the only non-vanishing form factor of $J^\dagger$ with two incoming particles; for example, there are non-vanishing overlaps between a state created by applying $J^\dagger$ on the a 2-electron state and states containing more quasi-particles than just a single electron. The additional quasi-particles can be visualized as (neutral) density waves or excitons, which are composed of a single electron and p quasi-holes. In the next section, we explore the selection rules that determine in a more general setting the possible out-states for which the form factor of $J^\dagger$ with a given in-state is non-vanishing.

4.5 A Form Factor Selection Rule

For the approximation of the electron or hole Green's functions on a $\nu = \frac{1}{p}$ fractional quantum Hall effect edge we can use a form factor expansion. An important step is then to establish which form factors can be excluded from the analysis from the start. We will show that such a selection rule can be formulated in the CS-basis. Using the mapping between the two bases the results can be translated to the fqH-basis.
The power of Jack Polynomial technology (JPT) in analyzing the formfactor

$$_N\langle\{\mu\}_J, q|J_{-m+q+p-\frac{p}{2}}|q+p, \{\nu\}_J\rangle_N \quad (4.74)$$

comes from the fact that JPT can be used twice, because a product of vertex-operators can be written as a sum over products of a 'coordinates' Jack-polynomials $P^p_{\{\lambda\}}(\{z_i\})$ and 'bosonic modes' Jack-operators $J^{\frac{1}{p}}_{\{\lambda\}}$. Both the coordinate and the bosonic mode Jack-polynomials can be manipulated using the results for Jack-polynomials in mathematical literature [76].
In section 4.4.3 we already made use of the freedom to rearrange the part of the normal ordered operator that survives after application of the normal ordered

operator to the vacuum. Here we will make use of this possibility again, and this will enable us to analyze the action of a single mode-operator $J_{-m+q+p-\frac{p}{2}}$ on a state created by a Jack-operator.
We can expand a product a product of $N+1$ edge-electron operators in Jack polynomials and Jack operators,

$$J(w)\prod_{i=1}^{N}J(z_i)|\tilde{q}\rangle =$$

$$w^{-\tilde{q}}\prod_{i=1}^{N}(w-z_i)^p\prod_{i<j}(z_i-z_j)^p\prod_{i=1}^{N}z_i^{-\tilde{q}}$$

$$\times\left(\sum_{\{\lambda\}}(-1)^{|\lambda|}P^p_{\{\lambda\}}(w)\,J^{\frac{1}{p}}_{\{\lambda'\}}\right)\left(\sum_{\{\rho\}}(-1)^{|\nu|}P^p_{\{\rho\}}(\{z_i\})\,J^{\frac{1}{p}}_{\{\rho'\}}\right)|\tilde{q}-pN-p,\{0\}\rangle. \tag{4.75}$$

From this expression we would like to extract information about the form factor ${}_N\langle\{\mu\}_J,q|J_{-m+q+p-\frac{p}{2}}|q+p,\{\nu\}_J\rangle_N$ where we now used normalized Jack-states

$$|q,\{\nu\}_J\rangle_N=(-1)^{|\nu|}\left(j^p_{\{\nu'\}}\right)^{\frac{1}{2}}J^{\frac{1}{p}}_{\{\nu'\}}|q,\{0\}\rangle. \tag{4.76}$$

The state $|\tilde{q}-Np,\{\nu\}_J\rangle_N$ with $N=l(\nu)$ can be extracted from the $\prod_{i=1}^{N}J(z_i)|\tilde{q},\{0\}\rangle$ state by applying the operator $O_{\{\nu\},\tilde{q}}(\{z_i\};p)$ to it,

$$O_{\{\nu\},\tilde{q}}(\{z_i\};p)=$$

$$(p^p_{\{\nu\},l(\nu)})^{-1}(j^{\frac{1}{p}}_{\{\nu\}})^{-\frac{1}{2}}\left(\prod_{i=1}^{l(\nu)}\oint\frac{dz_i}{2\pi i}\frac{1}{z_i^{-\tilde{q}+1}}\right)P^p_{\{\nu\}}\left(\{z_i^{-1}\}\right)\Delta^p\left(\{z_i^{-1}\}\right) \tag{4.77}$$

where we made use of the inner product (4.12) on coordinate dependent Jack-polynomials. The norm $p^p_{\{\nu\},n}$ of the Jack-polynomials will drop out of the final result. We can write

$$J_{-m+q+p-\frac{p}{2}}|\tilde{q}-pl(\nu),\{\nu\}_J\rangle_N=$$

$$M_{m-q-p}(w)O_{\{\nu\},\tilde{q}}(\{z_i\};p)\,J(w)\prod_{i=1}^{l(\{\nu\})}J(z_i),|\tilde{q},\{0\}\rangle \tag{4.78}$$

where

$$M_m(w)=\oint\frac{dw}{2\pi i}\frac{1}{w^{m+1}}. \tag{4.79}$$

Now we can apply the operators $M_{m-q-p}(w)$ and $O_{\{\nu\},\tilde{q}}(p;\{z_i\})$ to the expansion of $J(w)\prod_{i=1}^{N} J(z_i)|\tilde{q}\rangle$ in Jack polynomials. If we take the inner product of the expansion with ${}_N\langle\{\mu\}_J, q|$, where $q = \tilde{q} - (N+1)p$, we can argue using general results about Jack-polynomial inner products which form factors have a non-zero value. We first rewrite the product

$$\prod_{i=1}^{N}(w-z_i)^p = \sum_{\{n_i\}} C^p_{\{n_i\}} w^{pN} \prod_{i=1}^{N}\left(\frac{z_i}{w}\right)^{n_i} \tag{4.80}$$

where $n_i = 0, 1, \ldots, p$. We insert this into the expansion given in (4.75), and we use that $P^p_{\{\lambda\}}(w)$ is zero when not of the form $\{\lambda\} = (\lambda_1, 0, 0, ...)$. From this we can see for which $M_{m-p-q}(w)$ we get contributions to the state $J_{-m+q+p-\frac{p}{2}}|\{\nu\}_J, \tilde{q} - pN\rangle$. Looking at the powers of w appearing in the expansion we find the following condition,

$$pN + \lambda_1 + q + p = |n| + m + \tilde{q} \text{ or equivalently } \lambda_1 = |n| + m. \tag{4.81}$$

Varying $|n| = \sum_i n_i$ over all allowed values $|n| = 0, \ldots, pN$ we find that λ_1 has to satisfy the inequalities,

$$m \leq \lambda_1 \leq m + pN \tag{4.82}$$

which alternatively can be written as a restriction on m

$$\lambda_1 \geq m \geq \lambda_1 - Np. \tag{4.83}$$

The last condition is strange from the perspective taken so far where we looked at a state depending on a given m and an expansion containing a sum over $\{\lambda\}$. Later we will see that the restrictions coming from the fact that only $\{\lambda\} = (\lambda_1)$ contributes to the form factor can be translated into an extra condition on λ_1 coming from $\{\mu\}$ and $\{\nu\}$. With the help of the inequality above this condition on λ_1 can be translated into a condition on m.

We will write $\langle\{\mu\}_J|\{\lambda\}_J\{\nu\}_J\rangle_N$ for the inner product $\langle q, \{\mu\}_J|J^{\frac{1}{p}}_{\{\lambda'\}}|q, \{\nu\}_J\rangle_N$. To find for which λ_1 we find contributions to the form factor we use Proposition 5.3 from Stanley.

Proposition: *If the tableaus $\{\mu\}$, (λ_1), $\{\nu\}$ label Jack-operators associated with the same parameter $\beta = \phi, J$ then the inner product*

$$\langle\{\mu\}_\beta|(\lambda_1)_\beta\{\nu\}_\beta\rangle \neq 0 \tag{4.84}$$

if and only if $\mu \subseteq \nu$ and ν/μ is a horizontal λ_1-strip.

The relation $\mu \subseteq \nu$ indicates that for all i we have $\mu_i \leq \nu_i$. The skew tableaux ν/μ is the tableaux containing all boxes which are in the tableau ν but not in the tableau μ. If every column of the skew tableau contains at most one box it

is called a horizontal strip and if furthermore the total number of boxes in it is λ_1 it is called a horizontal λ_1-strip.
From this proposition we get the condition

$$0 \leq \lambda_1 \leq \mu_1 \tag{4.85}$$

and only for ρ with $l(\rho) = l(\mu)$ or $l(\rho) = l(\mu) - 1$ the inner product can be non-zero. Furthermore, combining the inequalities we find

$$\max(0, m) \leq \lambda_1 \leq \mu_1 \tag{4.86}$$

this implies that

$$\mu_1 < m \Rightarrow J^{\dagger}_{m-q-p+\frac{p}{2}} |q, \{\mu\}_J\rangle = 0. \tag{4.87}$$

We can use the results we obtained so far to analyze the terms in the following expression for the form factor

$$\begin{aligned} {}_N\langle q, \{\mu\}_J | J_{-m+q+p-\frac{p}{2}} | \{\nu\}_J, q+p\rangle_N = \\ \frac{(j^p_{\{\nu\}})^{\frac{1}{2}}}{p^p_{\{\nu\},l(\nu)}} \sum_{\{n_i\},\{\lambda_1\},\{\rho\}} \delta_{(m-\lambda_1+|n|)} C^p_{\{n\}} {}_N\langle \{\mu\}_J | \{\lambda_1\}\{\rho\}\rangle \\ \left(\prod_{i=1}^{N} M_0(z_i)\right) m_n(\{z_i\}) \Delta^p\left(\{z_i^{-1}\}\right) \Delta^p(\{z_i\}) P^p_{\{\rho\}}\left(\{z_i\}\right) P^p_{\{\nu\}}\left(\{z_i^{-1}\}\right) \end{aligned} \tag{4.88}$$

where the summations extend over $i, j = 1, \ldots, N$ and $n = n_1, \ldots, n_N$ with $n_i \in 0, 1, \ldots, p$.
This expression implies that for given $\{\mu\}$ the set of $\{\nu\}$'s is reduced because the restrictions put on $\{\rho\}$ by $\{\mu\}$ restrict the shape of the $\{\nu\}$'s to partitions containing $\{\tilde{\mu}\} = (\mu_2, \ldots, \mu_M)$ with $M = l(\{\mu\})$. Furthermore the tableaus $\{\nu\}$ are smaller in the sense of dominance ordering then a partition which contains $\{\tilde{\mu}\}$ and has at most p legs and $l(\{\mu\})$ arms. This result is a generalization of a zero-temperature result from Lesage, Pasquier, and Serban [55]. The statement is made more precise in the following theorem,

Theorem: *The state $|q+p, \{\nu\}_J\rangle$ can only give a non-zero form factor*

$$ {}_N\langle \{\mu\}_J, q | J_{-m+q+p-\frac{p}{2}} | q+p, \{\nu\}_J\rangle_N \neq 0 \tag{4.89}$$

if it satisfies the following conditions

a. $|\nu| + m = |\mu|$

b.
1. $\nu_j \geq \mu_{j+1}$ *for all* j
2. $\nu_i \leq p$ *for* $i > l(\{\mu\})$

c. $m + \sum_{i \geq l(\{\mu\})} \nu_i \leq \mu_1$

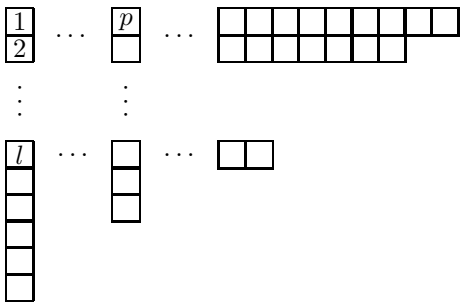

Figure 4.4: l arms and p legs

d. $\sum_{i=1}^{j} \nu_i \leq \sum_{i=1}^{j} (\mu_i + p)$

These conditions imply that the tableau $\{\nu\}$ *should have at most* p *legs and* $l(\mu)$ *arms, see fig. 4.4.*

(a.) This is a consequence of energy conservation and can be found from the product of three delta functions,

$$\delta_{|n|+m,\lambda_1} \delta_{\lambda_1+|\rho|,|\mu|} \delta_{|n|+|\rho|,|\nu|} \tag{4.90}$$

present implicitly in eq. (4.88).
(b.) Proposition 2.4 in [76] states that it is possible to rewrite any Jack polynomial as a linear combination of products of Jack polynomials labeled with horizontal strips $\mathcal{J}^p_{\{\lambda\}} = \prod_i P^p_{(\lambda_i)}$,

$$P^p_{\{\lambda\}} = \sum_{\{\sigma\} \geq \{\lambda\}} \tilde{q}_{\{\lambda\}\{\sigma\}} \mathcal{J}^p_{\{\sigma\}}. \tag{4.91}$$

An important difference between this expansion and the expansion of a Jack polynomial in monomial symmetric functions is that the sum runs over tableaus $\{\sigma\}$ satisfying $\{\sigma\} \geq \{\lambda\}$ instead of $\{\sigma\} \leq \{\lambda\}$. From repeated application of proposition 5.3 in [76] (see also section 4.3) to a product of two Jack polynomials, where one is expanded using the expansion above, it follows that the tableau labeling the non-expanded Jack polynomial is contained in every tableau labeling a Jack polynomial appearing in the product. Exchanging the roles we see that also the tableau labeling the other Jack polynomial is contained in these tableaus.
Combining this knowledge with the fact that by triangularity the monomial

symmetric functions $m_{\{n\}}$ can be expanded in Jack polynomials,

$$m_{\{n\}} = \sum_{\{\tau\}\leq\{n\}} \tilde{v}_{\{n\}\{\tau\}} P^p_{\{\tau\}}, \tag{4.92}$$

and applying this to the coordinate inner product in eq. (4.88) we find that $\{\rho\}$ is contained in $\{\nu\}$. The operator inner product shows that $\{\mu\}$ differs at most a horizontal strip from $\{\rho\}$ and thus $\{\rho\}$ contains $\{\tilde{\mu}\} = (\mu_2, \ldots, \mu_M)$. We can conclude that $\{\nu\}$ contains $\{\tilde{\mu}\}$ and we have obtained (b.1.).
We can extract extra information from examining this construction once more, under the addition of a horizontal strip the length of a column can grow with one box. If we now multiply the Jack polynomials $P^p_{\{tau\}}$ appearing in the expansion of the monomial symmetric function $m_{\{n\}}$ with the $\mathcal{J}^p_{\{\sigma\}}$ appearing in the expansion of the Jack polynomial $P^p_{\{\rho\}}$ we find that the maximal difference in column length between $\{\nu\}$ and $\{\tau\}$ is $l(\{\rho\})$ from which we can conclude that the only columns in $\{\nu\}$ which have a length exceeding $l(\{\mu\}) \geq l(\{\rho\})$ are those columns for which the corresponding tableau labeling the monomial symmetric function has a column of non-zero length. Because only monomial symmetric functions with at most p non-zero columns appear we obtain (b.2.).
(c.) This is a simple consequence of (a.) and (b.1.)
(d.) By definition the Jack polynomials can be expanded in monomial symmetric functions, so we have

$$P^p_{\{\rho\}} = \sum_{\{\sigma\}\leq\{\rho\}} v_{\{\rho\}\{\sigma\}} m_{\{\sigma\}}. \tag{4.93}$$

If we now use this expansion $P^p_{\{\rho\}}$ and then multiply the resulting $m_{\{\sigma\}}$ in the expansion by $m_{\{n\}}$, then the products in the expansion will be linear combinations of $m_{\{\sigma'\}}$ satisfying $\{\sigma'\} \leq \{\sigma\} + \{n\}$,

$$\begin{aligned} m_{\{n\}} P^p_{\{\rho\}} &= \sum_{\{\sigma\}\leq\{\rho\}} v_{\{\rho\}\{\sigma\}} m_{\{n\}} m_{\{\sigma\}} \\ &= \sum_{\{\sigma'\}\leq\{\rho\}+\{n\}} u_{\{n\}\{\rho\}\{\sigma'\}} m_{\{\sigma'\}}. \end{aligned} \tag{4.94}$$

Expanding the $m_{\{\sigma'\}}$ as we did in the proof of (b.) the sum over products of monomial symmetric functions can be rewritten in terms of Jack polynomials again,

$$m_{\{n\}} P^p_{\{\rho\}} = \sum_{\{\sigma\}\leq\{\rho\}+\{n\}} w_{\{n\}\{\rho\}\{\sigma\}} P^p_{\{\sigma\}} \tag{4.95}$$

and we find that $\{\nu\}$ is smaller in the sense of dominance order than a tableau $\{\sigma\}$ of the form $\{\rho\} + \{n\}$. Since $n_i \leq p$ and $\rho_i \leq \mu_i$, the result (d.1) follows.

5

Finite Temperature Green's Functions

5.1 Fermi-liquid to fqH-edge tunneling

Having checked that the thermodynamics of fqH-edges is correctly reproduced in the new quasi-particle language and having established some of the form factors properties we are now ready to move on and consider transport properties. Following the set-up of a number of experiments, we shall consider a situation where electrons (or holes) from a Fermi-liquid reservoir are allowed to tunnel into a $\nu = \frac{1}{p}$ fqH-edge. The DC IV-characteristic for this set-up, which were first computed by Kane and Fisher [46] (see also [82, 18]), show a cross-over from a linear (thermal) regime into a power-law behavior at high voltages and thus present a clear fingerprint of the Luttinger liquid features of the fqH-edge. The experimental results from [19] are in agreement with these predictions.
The calculations by Kane and Fisher were based on bosonization and on the Keldysh formalism for non-equilibrium transport. Our goal here is to see if we can reproduce their results in an approach directly based on the edge quasi-particle formalism. Before going into this, we would like to stress that the 'Thermodynamic Bethe Ansatz (TBA) quasi-particles' behind the approach of [18] are quite different from what we have here, the most important distinction being that the TBA quasi-particles are a combination of degrees of freedom of both sides of the tunneling barrier; they do not exist for a $\nu = \frac{1}{p}$ edge in isolation.
If the $\nu = \frac{1}{p}$ fqH-edge were to behave as a Fermi-liquid, we could calculate charge transport across a barrier using a simple (Boltzmann) kinetic equation of the form

$$I(V,T) \propto e \int_{-\infty}^{\infty} d\epsilon\, W \left\{ f_1(\epsilon - eV) F_2(\epsilon) - F_1(\epsilon - eV) f_2(\epsilon) \right\} \ , \tag{5.1}$$

with $f(\epsilon)$ and $F(\epsilon)$ the Fermi-Dirac distributions for electrons and holes, respectively, and W the probability for an electron or hole of energy ϵ to cross the barrier and enter the edge. As is well known, this Boltzmann equation leads to an ohmic and temperature-independent current. Now that we have seen that the non-Fermi liquid features of the $\frac{1}{p}$ edge can be captured via the statistics of the edge quasi-particles we can try to write a 'Boltzmann equation' for transport to and from fqH-edges by putting in appropriate generalizations $g(\epsilon)$ and

$G(\epsilon)$ of the quantities $f_2(\epsilon)$ and $F_2(\epsilon)$, respectively. Before giving precise results we shall consider a 'naive' expression based on the intuition from the quasi-particle approach. In first approximation, the factor $g(\epsilon)$, which describes the probability for an electron to leave a $\nu = \frac{1}{p}$ edge, comprises two effects

1. a correlation effect, which can be traced to the non-trivial scaling dimension of the edge electron operator. At zero temperature, this is the so-called tunneling density of states

$$A^+(\epsilon) \propto \epsilon^{p-1} \ , \tag{5.2}$$

2. a temperature dependence related to the exclusion statistics properties of the edge electrons. As we have seen, the natural factor associated to the presence of an edge electron is the distribution function

$$\bar{n}_{g=p}(\epsilon) \ . \tag{5.3}$$

Combining these factors, we come to the naive expressions

$$\tilde{g}(\epsilon) = \epsilon^{p-1}\, \bar{n}_{g=p}(\epsilon) \ , \tag{5.4}$$

and by similar reasoning we obtain

$$\tilde{G}(\epsilon) = \epsilon^{p-1}\, e^{\beta\epsilon}\, \bar{n}_{g=p}(\epsilon) \ , \tag{5.5}$$

where the thermal factor $e^{\beta\epsilon}\, \bar{n}_p(\epsilon)$ has been dictated by the requirement of detailed balance or outside the context of tunneling by the KMS-condition.
One quickly finds that the Boltzmann equation with factors $\tilde{g}$ and $\tilde{G}$ is not exact at finite temperature. Later we shall further comment on this equation and argue that it can be viewed as part of a first stage in a systematic approach. Before we come to that, we shall in the next section present a particularly simple derivation of the exact perturbative I-V characteristics for tunneling from a Fermi-liquid to a $\nu = \frac{1}{3}$ fqH-edge. This derivation uses the idea of a kinetic equation, together with the algebraic properties of the edge electrons.

5.1.1 Kinetic Equation for Inter Edge Transport

A careful derivation, based directly on the form of the tunneling Hamiltonian

$$H_{int} \propto t \int d\epsilon \, \left[\Psi^\dagger_{\nu=1}(\epsilon) \Psi_{\nu=\frac{1}{3}}(\epsilon) + \text{h.c.} \right] \ , \tag{5.6}$$

where $\Psi^\dagger_\nu$ denotes the electron creation operator for a filling fraction ν fqH-edge, leads to the following kinetic equation, see e.g. [82],

$$I(V,T) \propto e\, t^2 \int_{-\infty}^{\infty} d\epsilon \, [f(\epsilon - eV) G(\epsilon) - F(\epsilon - eV) g(\epsilon)] \ , \tag{5.7}$$

where G, g are one particle Green's functions

$$g_V(\epsilon) = \langle \Psi^\dagger_{\nu=\frac{1}{3}}(\epsilon)\Psi_{\nu=\frac{1}{3}}(\epsilon)\rangle_{V,T} \ , \qquad G_V(\epsilon) = \langle \Psi_{\nu=\frac{1}{3}}(\epsilon)\Psi^\dagger_{\nu=\frac{1}{3}}(\epsilon)\rangle_{V,T} \tag{5.8}$$

for edge electrons in the $\nu = \frac{1}{3}$ fqH-edge, taken at $V = 0$. Note that the expression 5.7 is perturbative as it gives the lowest non-trivial order in the parameter t.
The quantities $G_V(\epsilon)$ and $g_V(\epsilon)$ can be determined by using two simple observations. The first is that of detailed balance, which can be phrased as the requirement that at zero voltage there should be no current flowing. This fixes the ratio of $G_V(\epsilon)$ and $g_V(\epsilon)$ according to

$$G_V(\epsilon) = e^{\beta(\epsilon - eV)} g_V(\epsilon) \ . \tag{5.9}$$

The second observation uses the algebraic properties of the edge electron operator, which include the anti-commutation relation

$$\left\{ \Psi^\dagger_{\nu=\frac{1}{3}}(\epsilon), \Psi_{\nu=\frac{1}{3}}(\epsilon) \right\} = \frac{2\pi}{L}\frac{1}{\rho_0}\epsilon^2 + 6\frac{E}{\rho_0} + 6\epsilon\frac{\Delta Q}{e\rho_0} \ . \tag{5.10}$$

In this formula, E is the operator for the total energy per unit length, and ΔQ is the operator for the total charge per unit length. Clearly, this anti-commutator fixes the sum $G_V(\epsilon)+g_V(\epsilon)$. The expectation values of energy and charge follow directly from our analysis in chapter 3. We find

$$\langle E \rangle = \rho_0 \left(\frac{\pi^2}{6\beta^2} + \frac{(eV)^2}{6} \right) \ , \qquad \langle \Delta Q \rangle = -e\rho_0 \frac{(eV)}{3} \tag{5.11}$$

and obtain the exact expressions

$$G_V(\epsilon) = \frac{(\epsilon - eV)^2 + \frac{\pi^2}{\beta^2}}{e^{-\beta(\epsilon - eV)} + 1} \ , \quad g_V(\epsilon) = \frac{(\epsilon - eV)^2 + \frac{\pi^2}{\beta^2}}{1 + e^{\beta(\epsilon - eV)}} \ . \tag{5.12}$$

They lead to I-V characteristics

$$I(V,T) \propto e\, t^2\, \beta^{-3} \left(\frac{\beta eV}{2\pi} + \left(\frac{\beta eV}{2\pi} \right)^3 \right) \ , \tag{5.13}$$

in agreement with the result obtained in different approaches [46, 18].
Clearly, the Green's functions 5.8 can be evaluated in other ways, for example by using a conformal transformation in the x, t domain [82]. We would like to stress that our derivation is more direct and uses nothing more than the fundamental anti-commutation relation of the edge electrons. For $\nu = \frac{1}{3}$, these are particularly simple as they derive from the so-called $N = 2$ super-conformal algebra, which has been well- studied in other contexts. For other filling fractions the fundamental anti-commutators look more complicated but are available in principle.

5.1.2 Interpretation in Terms of Exclusion Statistics

If we compare the exact kinetic equation for $\nu = \frac{1}{3}$ with a naive generalized Boltzmann equation, we see that the mistake in the latter is in the approximation of the Green's function $g(\epsilon)$ by a the product $\tilde{g}(\epsilon)$ of a tunneling density of states times a Haldane distribution for fractional statistics.
The reason why this approximation turns out to be rather poor is that the operator $N_\Psi(\epsilon) = \Psi^\dagger_{\nu=\frac{1}{3}}(\epsilon)\Psi_{\nu=\frac{1}{3}}(\epsilon)$ inside a fqH-edge is not to be viewed as a simple counting operator weighted by the appropriate power law of ϵ. This fact can be traced to the non-trivial operator terms in the r.h.s. of eq. 5.10. To further illustrate this point we evaluated the expectation value of the operator $N_\Psi(\epsilon)$ in a normalized one-electron state $|\epsilon'\rangle$

$$\langle\epsilon'|N_\Psi(\epsilon)|\epsilon'\rangle \propto \epsilon^2\delta(\epsilon-\epsilon') + 6\frac{(\epsilon'-\epsilon)(\epsilon'^2+\epsilon^2)}{\epsilon'^2}\Theta(\epsilon'-\epsilon)\ . \tag{5.14}$$

This result shows an interaction effect in the action of $N_\Psi(\epsilon)$ on a one-electron state: rather than just counting quanta of energy ϵ, the operator $N_\Psi(\epsilon)$ is sensitive to the presence of quanta at energy $\epsilon' > \epsilon$ as well. In the Green's function $g(\epsilon)$ (for $\epsilon > 0$), the first term on the r.h.s. of eq. 5.14 corresponds to $\tilde{g}(\epsilon)$, while the second term leads to the following correction term

$$g^{(1,0)}(\epsilon) - \tilde{g}(\epsilon) = 6\int_\epsilon^\infty d\epsilon'\,\frac{(\epsilon'-\epsilon)(\epsilon'^2+\epsilon^2)}{\epsilon'^2}\,\bar{n}_3(\epsilon')\ . \tag{5.15}$$

In figure 5.1 we have plotted the exact result for $g(\epsilon)$ against the approximations $\tilde{g}(\epsilon)$ and $g^{(1,0)}(\epsilon)$. Clearly, the correction term included in $g^{(1,0)}(\epsilon)$ greatly improves the accuracy of the description.
The situation here can be described as follows. As far as thermodynamics goes, the distribution functions $\bar{n}_3(\epsilon)$ and $\bar{n}_{\frac{1}{3}}(\epsilon)$ give exact results for quantities such as specific heat and conductances. However, the operators $\Psi^\dagger_{\nu=\frac{1}{3}}(\epsilon)$, $\Psi_{\nu=\frac{1}{3}}(\epsilon)$ are not one-particle operators in the usual sense, as they do not simply add or extract a single quasi-particle from a many-particle state. In edge tunneling experiments, the edge system communicates with a Fermi liquid via the operators $\Psi^\dagger_{\nu=\frac{1}{3}}(\epsilon)$ and $\Psi_{\nu=\frac{1}{3}}(\epsilon)$ and we can not avoid interaction effects. We do believe, however, that a systematic expansion based on the quasi-particle picture is possible. Evidence for this claim is the calculation above and the calculation following in the rest of this chapter.

5.2 Form Factor Expansion

The Green's function $g(\epsilon)$ can be viewed as a one-point function for the operator $N_\Psi(\epsilon) = \Psi^\dagger_{\nu=\frac{1}{p}}(\epsilon)\Psi_{\nu=\frac{1}{p}}(\epsilon)$. In the formulation on the finite system of size L, this operator is represented as $N_J(m) = aJ_{-m}J^\dagger_m$, with $\epsilon = am$, with $a = \frac{2\pi}{L\rho_0}$ the energy level spacing in the finite size system. This one-point function is

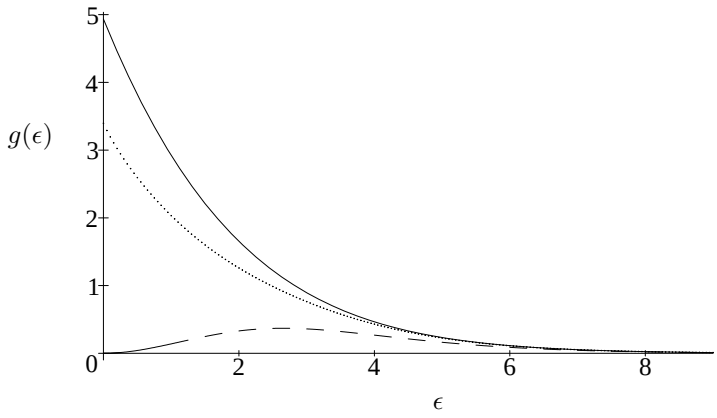

Figure 5.1: One-particle Green's function $g(\epsilon)$ in units $[\beta]$ as a function of the energie ϵ in units $[\beta^{-1}]$ for filling fraction $\nu = \frac{1}{3}$ and at zero voltage. The drawn curve is the exact result 5.12; the dashed curve is the approximation $\tilde{g}(\epsilon)$ and the dotted curve corresponds to $g^{(1,0)}(\epsilon)$.

formally expressed as

$$\frac{\sum_{\text{state}\in\mathcal{H}}\langle\text{state}|N_J(m)|\text{state}\rangle\exp(-\beta E_{\text{state}})}{\sum_{\text{state}\in\mathcal{H}}\langle\text{state}|\text{state}\rangle\exp(-\beta E_{\text{state}})}. \tag{5.16}$$

The sum runs over a basis of the full Hilbert space of the edge CFT, and we can opt for the fqH quasi-particle basis discussed in chapter 4. The idea is now that the matrix elements $\langle\text{state}|N_J(m)|\text{state}\rangle$ are dominated by processes where only a few of the quasi-particles that are present in a concrete basis state $|\{m_i;n_j\}\rangle$ participate.
For the case at hand, the lowest contribution comes from 1-particle states $|(m_1)\rangle$, for which one computes the matrix element

$$D^{(1,0)}(m;m_1) = {}_N\langle(m_1)|J_{-1-m}J^\dagger_{+1+m}|(m_1)\rangle_N. \tag{5.17}$$

The expected presence of an edge electron of energy m_1 is given by the distribution function $n_p(\epsilon_1 = am_1)$. This leads to the following contribution to the Green's function

$$g^{(1,0)}(\epsilon) = a\sum_{m_1} D^{(1,0)}(m,m_1)\bar{n}_p(am_1). \tag{5.18}$$

If we now consider the matrix element of $N_J(m)$ against a two-electron state, we find (see next subsection) that it is not simply the sum of two 1-particle contributions. The left-over part is what we call the irreducible 2-electron matrix element

$$D^{(2,0)}(m;m_1,m_2) = {}_N\langle(m_1,m_2)|J_{-3-m}J^\dagger_{+3+m}|(m_2,m_1)\rangle_N$$

$$-{}_N\langle(m_1)|J_{-3-m}J^\dagger_{+3+m}|(m_1)\rangle_N - {}_N\langle(m_2)|J_{-3-m}J^\dagger_{+3+m}|(m_2)\rangle_N \ . \tag{5.19}$$

It leads to an additional contribution $g^{(2,0)}(m)$ to the Green's function

$$g^{(2,0)}(\epsilon) = a\sum_{m_1,m_2} D^{(2,0)}(m;m_1,m_2)\bar{n}_p(am_1)\bar{n}_p(am_2) \ . \tag{5.20}$$

Similarly, we define

$$D^{(1,1)}(m;m_1,n_1) = {}_N\langle(n_1,m_1)|J_{-1-m}J^\dagger_{+1+m}|(m_1,n_1)\rangle_N - {}_N\langle(m_1)|J_{-1-m}J^\dagger_{+1+m}|(m_1)\rangle_N \tag{5.21}$$

and

$$g^{(1,1)}(\epsilon) = a\sum_{m_1,n_1} D^{(1,1)}(m;m_1,n_1)\bar{n}_p(am_1)\bar{n}_{\frac{1}{p}}(an_1) \tag{5.22}$$

Continuing in this manner, we build up the following expansion

$$g = \sum_{M,N} g^{(M,N)}(\epsilon)$$
$$g^{(M,N)}(\epsilon) = a\sum_{\{m_i;n_j\}} D^{(M,N)}(m;\{m_i;n_j\})\prod_i \bar{n}_p(am_i)\prod_j \bar{n}_{\frac{1}{p}}(an_j) \tag{5.23}$$

We remark that an expansion of precisely this type has been proposed by LeClair and Mussardo [54], see also Saleur [68]. This work was done in the context of integrable qft's, that are fully characterized by a factorized S-matrix. In such a context, the irreducible form factors are constrained by the form factor axioms, and the distribution functions have their origin in a TBA procedure. Although clearly in the same spirit, the analysis that we present here is very different at the technical level. We obtain the relevant form factor by explicit computation in a theory that is regularized by the finite size of the fqH edge, and we have identified the relevant distribution functions by analyzing the state counting of the (discrete) spectrum of the finite-size system. We thus do not rely on an underlying (massless) S-matrix point of view.

5.3 Finite T Green's Function for $p = 2$

As a proto-type study for a form factor expansion based on CFT quasi-particles, we now analyze the Green's function $g(\epsilon)$, for $p = 2$ in that spirit. Obviously, an exact result is easily obtained

$$g(\epsilon) = \frac{\epsilon}{e^{\beta\epsilon} - 1} \ . \tag{5.24}$$

The Bose-Einstein denominator in this expression has its origin in the fact that the operators J, $J^\dagger$ satisfy bosonic commutation relations. In the spirit of the quasi-particle formulation, we wish to treat the J, $J^\dagger$-quanta as quasi-particles with exclusion statistics $g = 2$, and see if we can recover the Green's function $g(\epsilon)$ in such an approach.
To evaluate explicitly the leading terms in the form factor expansion 5.23 for $g(\epsilon)$, we need to evaluate the relevant irreducible matrix elements. While it is clear that these matrix elements have very special mathematical properties, we here compute them by a simple brute force computation, relying on the algebraic properties of the operators $J^\pm$, Q and $\phi^\pm$, which form a representation and a doublet representation of $SU(2)_1$,

$$\begin{aligned}
[J_s^+, J_r^-] &= s\delta_{s+r} + Q_{s+r} \\
[Q_s, Q_r] &= 2s\delta_{s+r} \\
[Q_s, \phi_r^\pm] &= \pm\phi_{s+r}^\pm \\
[Q_s, J_r^\pm] &= \pm 2J_{s+r}^\pm \\
[J_s^\mp, \phi_r^\pm] &= \phi_{s+r}^\mp \\
[J_s^\pm, \phi_r^\pm] &= 0
\end{aligned}$$

. (5.25)

Furthermore, we use the explicit form of the two-particle states $|(m_2, m_1)\rangle$ and $|(m_1; n_1)\rangle$ at $p = 2$,

$$|(m_2, m_1)\rangle = |m_2, m_1\rangle + \frac{2}{m_2 - m_1 + 3} \sum_{l>0} |m_2 + l, m_1 - l\rangle, \tag{5.26}$$

and

$$|(m_1; n_1)\rangle = |m_1; n_1\rangle - \frac{1}{(m_1 + 2n_1 + 1)} \sum_{l>0} |m_1 + l; n_1 - l\rangle, \tag{5.27}$$

with

$$|m_2, m_1\rangle = J_{-3-m_2} J_{-1-m_1} |0\rangle \ , \qquad |m_1; n_1\rangle = J_{-1-m_1} \phi_{-\frac{1}{4}-n_1} |0\rangle \ . \tag{5.28}$$

These states have the following norms

$$N_{(m_2,m_1)} = \langle (m_1, m_2) | (m_2, m_1)\rangle = \frac{m_2 - m_1 + 1}{m_2 - m_1 + 3} (m_2 + 3)(m_1 + 1), \tag{5.29}$$

and

$$N_{(m_1;n_1)} = \langle (n_1; m_1) | (m_1; n_1)\rangle = \frac{m_1 + 2n_1 + 2}{m_1 + 2n_1 + 1} (m_1 + 1) C_{n_1}^{(-\frac{1}{p})} \ . \tag{5.30}$$

One electron

For the (irreducible) one electron matrix element we found

$$D^{(1,0)}(m;m_1) = {}_N\langle(m_1)|J_{-1-m}J^\dagger_{+1+m}|(m_1)\rangle_N = \\ (m+1)\delta_{m,m_1} + 2\left(1-\frac{m+1}{m_1+1}\right)\Theta(m<m_1)\ . \tag{5.31}$$

Two electrons

For the irreducible two electron matrix element we find

$$\begin{aligned} D^{(2,0)}(m;m_2,m_1) = & \\ & \delta_{m-m_2}\frac{-2(m_2+3)}{m_2-m_1+3} + \delta_{m-m_1+2}\frac{-2(m_1+1)}{m_2-m_1+1} \\ & +\frac{4}{(m_2-m_1+3)}\frac{1}{(m_2-m_1+1)}\frac{1}{(m_1+1)(m_2+3)} \\ & \times[\Theta(m<m_1-2)\,P(m;m_1,m_2) \\ & +\Theta(m<m_2<m+m_1)\,Q(m;m_1,m_2) \\ & +\Theta(m<m_2)\,R(m;m_1,m_2)]\ , \end{aligned} \tag{5.32}$$

with

$$\begin{aligned} P(m;m_1,m_2) = & \\ & (m_2-m_1+3)(m_1-m-2)(2m_1-m_2-3) \\ & +(m_1-m-2)(m_1-m-3)(-3m_2+\frac{5}{3}m_1+\frac{1}{3}m-\frac{26}{3}) \\ & +(m+3)[-2(m_2-m_1+3)(2m_1-m-1) \\ & -2m_1(m_1+1)+(m+3)(m_2+m_1-m+1)] \\ Q(m;m_1,m_2) = & \\ & (m_1-m_2+m+1)[(m_2-m_1+3)^2+2(m_2-m_1+3)(m_1-m_2+m) \\ & +\frac{2}{3}(m_1-m_2+m)(m_1-m_2+m-1)] \\ R(m;m_1,m_2) = & \\ & (m_2-m)(m_1+1)(m_2-m_1+3)+\frac{1}{3}m_1(m_1+1)(m_1+3m_2-3m+2) \end{aligned} \tag{5.33}$$

The polynomials P, Q and R enjoy special properties, which include

$$(P+Q+R)(m;m_1,m_2) = \frac{1}{3}(m_1-m_2-1)(m_1-m_2-2)(m_1-m_2-3)\ . \quad (5.34)$$

One electron and one quasi-hole

The irreducible matrix element with one electron and one hole is found to be

$$\begin{aligned} D^{(1,1)}(m;m_1,n_1) = & \\ & \delta_{m_1,m}\,\frac{m_1+1}{m_1+2n_1+1} \\ & +\Theta(m<m_1)\frac{1}{C_{n_1}^{(-\frac{1}{2})}(m_1+2n_1+2)(m_1+2n_1+1)(m_1+1)} \\ & \quad\times\left[C_{n_1-m_1+m}^{(-\frac{1}{2})}\,S(m;m_1,n_1)+C_{n_1}^{(-\frac{1}{2})}\,T(m;m_1,n_1)\right]\ , \end{aligned} \quad (5.35)$$

with

$$\begin{aligned} S(m;m_1,n_1) = & \\ & (m_1+2n_1+1)^2+(m+n_1-m_1)(\frac{8}{3}-4(m_1+2n_1+2))+\frac{4}{3}(m+n_1-m_1)^2 \end{aligned}$$

$$\begin{aligned} T(m;m_1,n_1) = & \\ & 2(m_1-m)((m_1+2n_1+1)^2-1)+2(2n_1+1)(m_1-m-1) \\ & +2(\frac{2}{3}n_1+1)(2n_1+1)\ . \end{aligned} \quad (5.36)$$

5.3.1 Evaluating the Series

With the information collected in the previous subsections, we can evaluate the 1-particle and 2-particle contributions $g^{(1,0)}$, $g^{(2,0)}$ and $g^{(1,1)}$ to the Green's function $g(\epsilon)$.

The expressions 5.18, 5.20, and 5.22 for $g^{(2,0)}$ and $g^{(1,1)}$ are discrete sums, which we wish to study in the limit $a \to 0$. In this limit, one may view the expressions as Riemann sums and evaluate them using continuous integrals; however, one needs to be careful because the integrands as they stand have singularities, and the sums are not term-by-term convergent. One may check however that by carefully redistributing some of the terms, one obtains convergent sums that can be approximated by the corresponding continuous integrals. Proceeding in this manner, and using a numerical integrator, we obtained the results plotted in figure 5.3 and figure 5.2 .

We observe that the form factor series converge in the following sense: while the 1-particle terms agree with the exact result for ϵ greater than about $3k_BT$, the result with 2-particle terms included has reached the exact value at ϵ greater

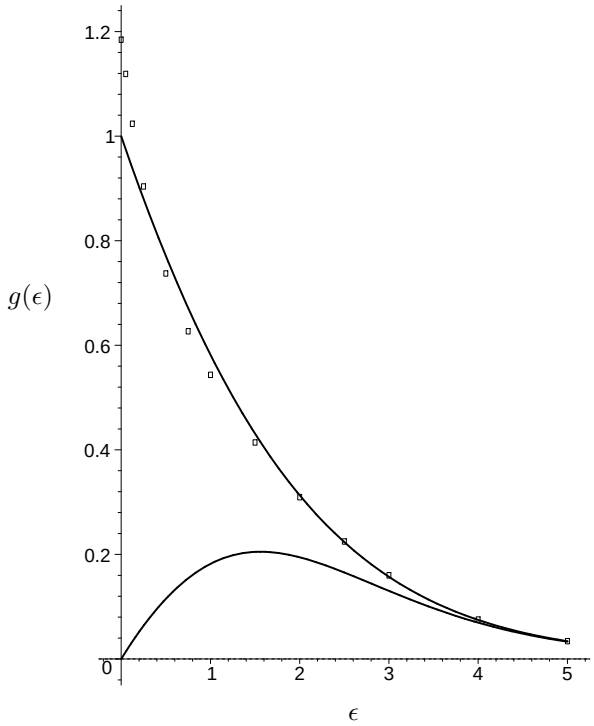

Figure 5.2: One-particle Green's function $g(\epsilon)$ in units $[\beta = \frac{1}{k_B T}]$ for filling fraction $\nu' = \frac{1}{2}$ as a function of energy in units $[\beta^{-1}]$. The upper solid curve is the exact result, the data points represent the sum of all contributions with up to two particles present. The lower solid curve represents the naive approximation discussed in the introduction of this chapter.

than about $2k_BT$. For energies $\epsilon \ll k_BT$, the thermal factors do not efficiently suppress many particle contributions, and the convergence of the form factor expansion is expected to be slow.

We remark that the asymptotic behavior for $\epsilon \gg k_B T$ of the 2-particle terms is

$$g^{(2,0)}(\epsilon) \sim c_2 e^{-\beta\epsilon} \qquad g^{(1,1)}(\epsilon) \sim c_{\frac{1}{2}} e^{-\beta\epsilon} \tag{5.37}$$

with

$$c_2 = -2\int_0^\infty d\epsilon_1 \bar{n}_2(\epsilon_1) \ , \qquad c_{\frac{1}{2}} = \int_0^\infty d\tilde{\epsilon}_1 \bar{n}_{\frac{1}{2}}(\tilde{\epsilon}_1) \ , \tag{5.38}$$

Remarkably, the duality relation 3.26 between the distributions leads to the relation

$$c_2 = -c_{\frac{1}{2}} \tag{5.39}$$

meaning that the Boltzmann tails of the 2-particle terms precisely cancel. This 'conspiracy' was needed as, numerically, it is seen that the deviation between the exact curve $g(\epsilon)$ and the 1-particle term $g^{(1,0)}(\epsilon)$ is far smaller than the individual Boltzmann tails of $g^{(2,0}$ and $g^{(1,1)}$.

5.4 Conclusions

The results in this chapter show that with the fqH-basis we identified the basis in which the one-point Green's function can be approximated very efficiently using only form factors which involve a few quasi-particles. From our results in the preceding chapter we can see that in an alternative basis, like the CS-basis, the number of quasi-particles needed to describe the same form factors ranges from a few for low energy states to many for the high energy states. This shows that the fqH-basis gives the most efficient description of the degrees of freedom relevant for the description of the one point Green's functions on the fqH-edge.

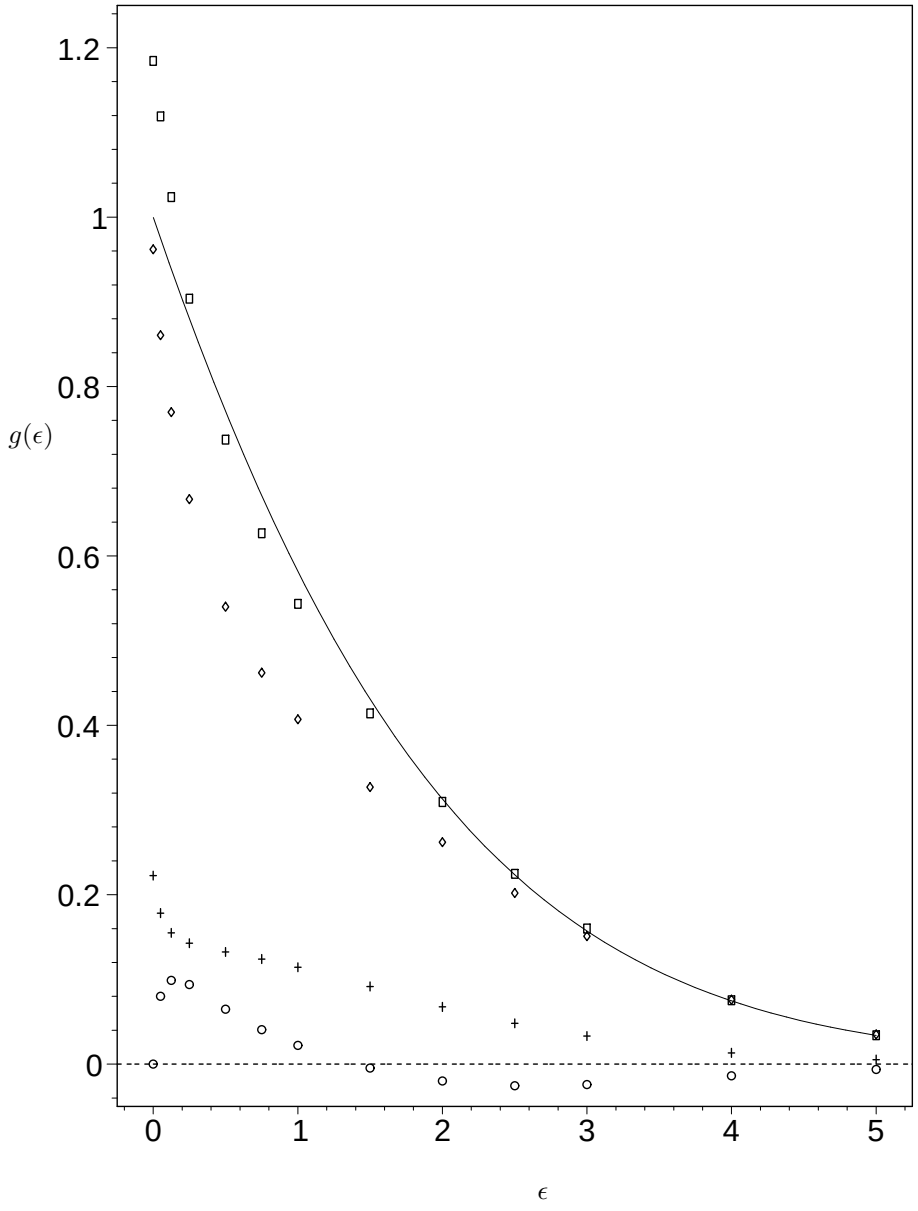

Figure 5.3: One-particle Green's function $g(\epsilon)$ for filling fraction $\nu' = \frac{1}{2}$ as a function of energy, both in units $[k_B T]$. The upper solid curve is the exact result; the data points are the numerical results for: $g^{(1,0)}$ (diamonds), $g^{(2,0)}$ (circles) and $g^{(1,1)}$ (crosses). The sum of all contributions is represented by squares.

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Summary

In this thesis we give a quasi-particle description of fractional quantum Hall systems. The idea behind such an approach is that an appropriate choice of quasi-particles can lead to a considerable simplification of the calculation of the physical properties of a system. More importantly, the identification of the appropriate quasi-particles often reflects a real understanding of the physics of a system. This summary will help the reader understand some of the ideas and the results which appear in this thesis.

S. 1 The Concepts in the Title

Most people are unfamiliar with the concepts in title of this thesis. In this section we introduce these concepts: fractional quantum Hall systems and quasi-particles. It should give enough insight to understand the next section in which we introduce some of the properties of quasi-particles in fractional quantum Hall systems.

S. 1.1 Particles and Quasi-particles

What is a quasi-particle? A quasi-particle is a long-lived pattern in matter which, at short time scales, shows the behavior of a particle. Although in general a physicist would not think this way, a grain of sugar can be thought of as a quasi-particle. It is clear that if we want to describe how sugar falls from a spoon it is more convenient to build our description on the roughly $1,000$ sugar grains than on the roughly $1,000,000,000,000,000,000,000$ sugar molecules. If we put the sugar in a cup of coffee and stir, we see that the sugar grains are no longer long-lived, and to understand how sugar dissolves in coffee we have to examine the properties of sugar molecules. This suggests that one should choose the quasi-particles appropriate to the process one tries to describe.
Another example of a quasi-particle is a wave on the ocean surface. This example is getting closer to a physicist's view on quasi-particles. In a wave the water molecules, the 'real' particles, move up and down. The wave itself, however, moves horizontally. A physicist calls this a collective effect: only if you have many particles, a collective, can you make a wave and this wave has new properties which can be very different from the properties of the individual particles. Another example is the wave in a football stadium. Now, supporters are the particles. They move up and down on their seats but the wave travels around the stadium.
An important property of particles is their statistics. The statistics of particles is a rule for combining particles. For the spoon of sugar we saw there were many grains on the spoon and in principle we could describe the sugar by giving the

position and orientation of every grain, assuming all the grains have the same size and shape. But we know we can use every position only once because there can not be two grains at the same place at the same time. This statistics is called fermionic and corresponds to Pauli's exclusion principle: every position can at the most be occupied by a single grain. For the waves on the ocean the statistics is called bosonic as different waves can occupy the same position at the same time: they can move through each other, and many small waves can combine into one big wave.

S. 1.2 Fractional Quantum Hall Systems

What is a fractional quantum Hall system? To answer this question we first present the devices (called heterostructures), which can be turned into fractional quantum Hall systems by cooling them to very low temperatures ($20\text{mK} - 4\text{K}$) and by putting them in a strong magnetic field.
A heterostructure is a sandwich of two semi-conductors. Semi-conductors form a class of materials that is very important to modern society because one can use them to build computers. The reason that all modern computers are built from semi-conductors is that their properties can be manipulated very well. A clean semi-conductor cannot conduct but if one adds the right kind of impurities, called electron acceptors or electron donors, the semi-conductor becomes conducting.
In a heterostructure the impurities, electron donors in this case, are put close to the interface between two different semi-conductors. The impurities donate electrons to the semi-conductor. In a normal semi-conductor these electrons would move through the semi-conductor, but in a heterostructure these electrons prefer to be close to the interface. If you cool the heterostructure to very low temperatures the electron movement is restricted to a thinner and thinner layer close to the interface. If the temperature is low enough the electrons will be frozen to the interface, but they can still move along the interface. Their world has become two-dimensional.
If the heterostructure is in a strong transverse magnetic field the electrons at the interface can show a new kind of behavior. This phenomenon was first observed in 1982 by Tsui, Störmer and Gossard. Soon after the first observation Laughlin gave an interpretation of the experiment. For this work Laughlin, Tsui and Störmer shared the 1998 Nobel-prize. What is this new behavior? Two figures will explain the phenomenon.
In figure S. 1.1 we see a four terminal geometry in which the fractional quantum Hall effect is measured. We look at the interface from above. To the interface six contacts are attached: the source (S), the drain (D) and four terminals. From the source electrons are injected into the sample and they leave the sample at the other side into the drain. The current is flowing in the opposite direction because the electrons have negative charge and the current direction refers to the movement of positive charge. The terminals are used to probe voltage differences. The voltage difference between terminal A and C is called the Hall-voltage V_H, and the voltage difference between A and B is called the longitudinal

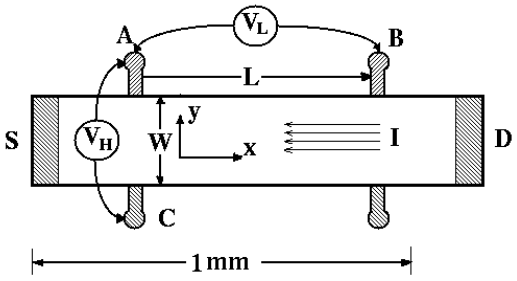


Figure S. 1.1: Four terminal geometry. Not drawn is the magnetic field, which is perpendicular to the two dimensional interface. Picture taken from [5].

voltage V_L.
When there is no magnetic field the Hall voltage will be zero. However, when a magnetic field B perpendicular to the interface is present and a current I is flowing there will be a non-zero Hall-voltage over the sample. It is the Lorentz force on a charge moving in a magnetic field which causes the Hall voltage. From the classical description of the Lorentz force we expect that the Hall voltage will behave as

$$V_H = C_{\text{Hall}} I B \tag{S. 1}$$

with C_{Hall} a constant known as the Hall coefficient. From this equation follows that the Hall-resistance

$$\rho_H = \frac{V_H}{I} \tag{S. 2}$$

will depend linearly on the magnetic field. However, in figure S. 1.2, in which the results of a measurement of ρ_H as a function of the magnetic field are depicted, we see that a structure with many plateaus appears instead of the predicted straight line. The plateaus occur when the the conductance $\sigma_H = \frac{1}{\rho_H}$ is at a value which is a simple odd fraction like $\frac{1}{3}$ or $\frac{2}{5}$ of the fundamental quantum of conductance $\frac{e^2}{h}$. The appearance of this structure is called the fractional quantum Hall effect. The name of the effect indicates that we need quantum mechanics instead of classical mechanics to understand it.
Now we can define fractional quantum Hall systems: The electrons at the interface in a heterostructure form a fractional quantum Hall system, when in a measurement of the Hall-resistance we are at the centre of a fractional quantum Hall effect plateau, with Hall conductance equal to a fractional (non-integer) multiple of $\frac{e^2}{h}$

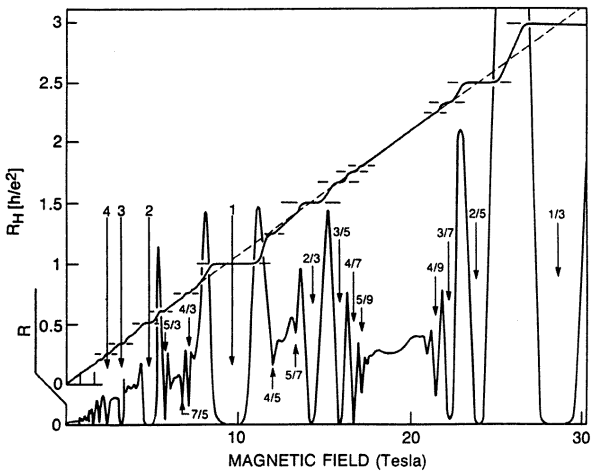


Figure S. 1.2: The Hall resistance ρ_H as a function of magnetic field B, picture taken from Willett et al.[83]

The occurence of the (fractional) quantum Hall effect, i.e., the formation of plateaus and transitions between them, has been explained by beautiful theories that combine the effects of long range interactions and disorder. In this thesis we do not discuss these aspects, but we focus entirely on the special states of matter that are formed in fractional quantum Hall systems.

S. 2 Properties of Quasi-Particles in fqH-Systems

The quasi-particles in fractional quantum Hall systems are special for two related reasons: they carry fractional charge and satisfy fractional statistics. In fact, an analogy to fractional charge and statistics can be found in a lot of construction toys as we will show in the first subsection.
Fractional quantum Hall systems are strongly correlated electron systems. Among other things this is visible in the way electrons leak from a fractional quantum Hall system to a metal needle in their vicinity. In the second subsection we will discuss the Green's function which plays a central role in describing the leak current.

S. 2.1 Fractional Charge and Statistics

Although the smallest free charges in the universe are believed to be the negative electron charge $-e$ and the positive proton charge $+e$, the quasi-particles in fractional quantum Hall systems carry a fractional charge. A simple example

shows that it is possible to make patterns which have a size equal to a fraction of the size of the system's building blocks.

Take Lego blocks with length three and height and width one and put them in a one layer thick row on a groundplate with width one and length $N \gg 1$. In this row there can be holes, i.e. unoccupied positions, which have a size equal to one-third of Lego block. If there are a few holes and many Lego blocks it is easier to describe the row by specifying the positions of the holes than by specifying all the positions of the blocks. In this situation a description in terms of quasi-particles (i.e. the holes) is shorter and more efficient. The holes which are the building blocks of our new description are 'fractional anti-blocks', and only if we have three of them in neighbouring positions is there enough room to replace them by a block. If our Lego blocks would carry charge $-e$ it would be natural to associate with every position a charge $+e/3$ which makes the system neutral if all positions are occupied. With this choice a hole acquires a charge $e/3$. This is the uncompensated background charge at an empty position.

The row of Lego blocks gives us a rough idea of fractional statistics or, more precisely, spectral shift statistics, which is the main topic of chapters 'Spectral Shift Statistics' and 'Quasi-Particle Thermodynamics'. Statistics of the blocks is directly related to the number of ways in which you can put M Lego blocks in a row on a ground plate of length N. The Fermi statistics we encountered before would correspond to Lego blocks of length 1 and the holes and the blocks would have the same size. Bose statistics would be slightly outside this picture because we have to allow for piling up the Lego blocks of length one in one position. Integer statistics coincides with Fermi statistics except that the size of the Lego blocks is an integer g which is larger than 1. Fractional statistics is obtained by switching from a description with blocks of size g to a description in terms of the holes and formulating the counting rules using the number of holes.

In the chapters we mentioned a lot of attention is paid to the concept of duality, which in fact is nothing but the statement we encountered before, namely, that to specify where the Lego blocks are is the same as specifying where the holes are. With some of oversimplification we can directly obtain the duality relation from the relations between the hole density and the Lego block density. If the ground plate has length N this length will be equal to the sum of the number of holes #(holes) plus the length of the blocks g multiplied by the number of blocks #(blocks),

$$N = \#(\text{holes}) + g\ \#(\text{blocks}). \tag{S. 3}$$

If we divide all terms by N, and call g #(holes)/N the hole density $\bar{n}$(holes), and #(blocks)/N the block density $\bar{n}$(blocks), we get

$$g^{-1}\bar{n}(\text{holes}) = 1 - g\bar{n}(\text{blocks}) \tag{S. 4}$$

which is similar to as the result obtained in equation 3.26 in this thesis. The normalization constant appearing in the definition of $\bar{n}$(holes) cannot be justified by this analogy. In a fractional quantum Hall system the constant can be related

to the energy difference between two neighboring holes, which instead of $1/g$ turns out to be $1/g^2$ of the difference between two neighboring electrons.
In the bulk of fractional quantum Hall systems the role of the length of the ground plate is played by the number of magnetic flux quanta and one electron occupies, for example, an area containing three flux quanta. If the whole two dimensional interface is filled with electrons it is possible to make three holes by removing one electron. It is also possible to make one hole by adding an extra flux quantum to the system. In our analogy adding a flux quantum would correspond to lengthening the groundplate with one without increasing the number of Lego blocks. For the edge of a fractional quantum Hall system there is no natural N but it turns out that the fractional statistics can be defined in small energy intervals that can be associated with a natural N.

S. 2.2 Removing Quasi-Particles

The interaction between a fractional quantum Hall system and the outside world can often be described in terms of adding and removing particles. An example of such an interaction could be the leakage of electrons from the system to a metal needle. To calculate the current from the system to the needle we need Green's function, which gives us (up to a constant) the number of electrons $g(\epsilon)\Delta\epsilon$ available for leaking at energies in the interval $[\epsilon, \epsilon + \Delta\epsilon)$. Furthermore, we need $W(\epsilon)$ which is the probability per unit time for a particle available in this energy interval to leak to the needle.
If we described the leakage from a metal system into the needle, $g(\epsilon)$ would simply be the expected occupation $\bar{n}(\epsilon)$ of a state of energy ϵ multiplied by the thermodynamic density of states $D(\epsilon)$,

$$g(\epsilon) = D(\epsilon)\bar{n}(\epsilon). \tag{S. 5}$$

The contribution to the current from electrons in this interval becomes

$$I_{\text{leakage}}([\epsilon, \epsilon + \Delta\epsilon)) \propto W(\epsilon)D(\epsilon)\bar{n}(\epsilon)\Delta\epsilon. \tag{S. 6}$$

The current will depend on the Green's function of the needle, but in our notation we suppressed that dependence to keep the argument simple.
In figure S. 2.3 we see the true Green's function and the approximation S. 5 of the Green's function for a specific fractional quantum Hall system. It is clear that the approximation fails in this case. The reason for this is that the electrons in the system strongly interact: it is as if they were connected to each other by strings, and if one pulls out one of the electrons many electrons follow. In reality the electrons repel each other but the effect is the same.
To express $g(\epsilon)$ in quantities like $D(\epsilon)$ and $\bar{n}(\epsilon)$ is the main goal of the work in the last chapters. Instead of $D(\epsilon)$ we need a set of matrix elements $D(\epsilon, \{\epsilon_i\}, \{\tilde{\epsilon}_j\})$ which describe processes in which the electron is available for leakage only after a reorganization of other quasi-particles present at energies $\{\epsilon_i\}, \{\tilde{\epsilon}_j\}$. In the simplest (fermionic) case only the quasi-particle put in or taken out takes part in the reorganisation the matrix element and $D(\epsilon)$ is the tunneling density of

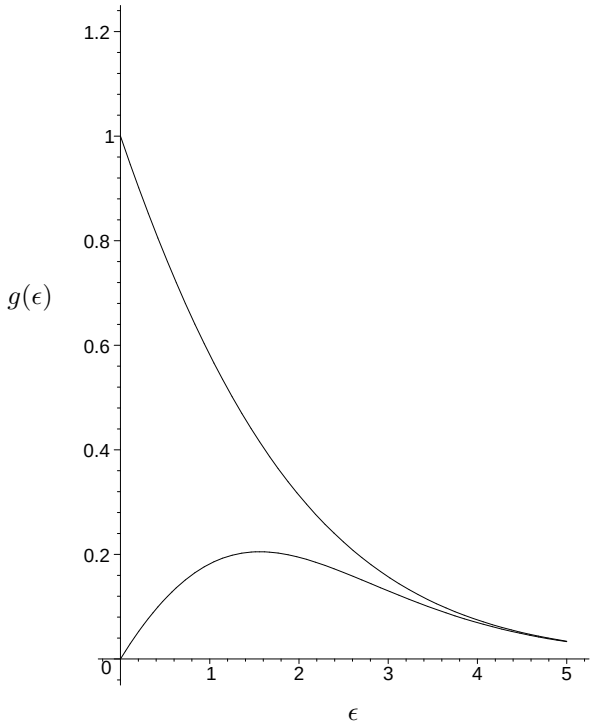


Figure S. 2.3: The naive approximation $D(\epsilon)\bar{n}(\epsilon)$ (lower curve) and the true Green's function $g(\epsilon)$ (upper curve) as a function of energy in units $[k_BT]$, for a specific fractional quantum Hall system.

states. Then we can $g(\epsilon)$ factorize into $D(\epsilon)\bar{n}(\epsilon)$. In general $g(\epsilon)$ is obtained by multiplying the matrix elements with the distribution functions $\bar{n}_{e^-}(\epsilon_i)$ and $\bar{n}_h(\tilde{\epsilon}_j)$ and summing over ϵ_i and $\tilde{\epsilon}_j$,

$$g(\epsilon) = \sum_{\{\epsilon_j\},\{\tilde{\epsilon}_i\}} D(\epsilon, \{\epsilon_i\}, \{\tilde{\epsilon}_j\})\bar{n}_{e^-}(\epsilon_1)\ldots\bar{n}_{e^-}(\epsilon_M)\bar{n}_h(\tilde{\epsilon}_1)\ldots\bar{n}_h(\tilde{\epsilon}_N). \qquad \text{(S. 7)}$$

In principle there are infinitely many of these matrix elements. Luckily it is expected that a very good approximation of $g(\epsilon)$ is obtained using matrix elements which describe processes in which only a few quasi-particles participate.

For the simplest fractional quantum Hall systems we were able to find strong evidence that this expectation is met; in figure S. 2.4 we plotted our results, which should be compared with the naive approximation in figure S. 2.3. Furthermore, we were able to develop a mathematical formalism needed for future generalizations to more complicated fractional quantum Hall systems or other effectively one-dimensional models.

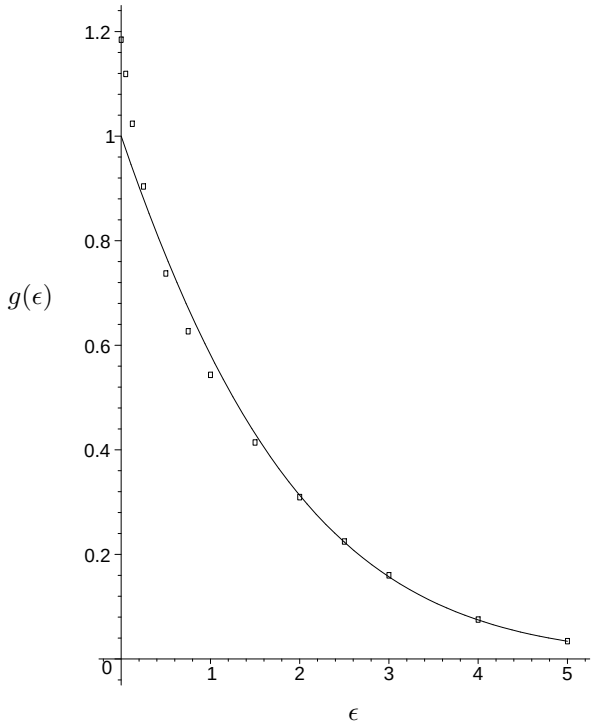


Figure S. 2.4: One-particle Green's function $g(\epsilon)$. The points represent our approximation and the solid curve is the exact result.

S. 3 Lessons Learned

In this thesis we established that the fractionally charged quasi-particles at the edge of a fractional quantum Hall system satisfy fractional spectral shift statistics. Furthermore, we found that it is possible to compute in good approximation quantities of experimental interest, using quasi-particle and quasi-hole distribution functions and matrix elements of suitable physical operators for the simplest fractional quantum Hall systems. Although a full understanding of this approximation is still lacking, we think we demonstrated its validity. For extension of the approach presented in this thesis to more complicated fractional quantum Hall systems or other low-dimensional strongly correlated systems further mathematical results in the field of Jack-polynomials or more general symmetric polynomials are needed. If such progress is made, the quasi-particle approach presented in this thesis can be further streamlined, and can be made into an important tool in the analysis of strongly correlated electrons.

Quasideeltjes voor Fractionele Quantum Hallsystemen

In dit proefschrift wordt een quasideeltjes beschrijving van fractionele quantum Hallsystemen geintroduceerd. De gedachte achter deze aanpak is dat een juiste keuze van quasideeltjes tot een sterke vereenvoudiging kan leiden bij de berekening van fysische grootheden. Bovendien is het vaak zo dat een goede keuze van de quasideeltjes leidt tot een beter begrip van de door deze deeltjes beschreven natuurkunde. Deze samenvatting zal enkele ideeën en resultaten die in dit proefschrift een belangrijke rol spelen introduceren bij de lezer.

N. 1 Betekenis van de Titel

Voor de meeste mensen zijn de woorden in de titel van dit proefschrift nieuw. Daarom introduceren we in dit deel van deze samenvatting deze begrippen, nl. *quasideeltjes* en *fractionele quantum Hallsystemen*. Hierna is het hoop ik zo dat de discussie over de eigenschappen van de quasideeltjes te begrijpen is.

N. 1.1 Deeltjes en Quasideeltjes

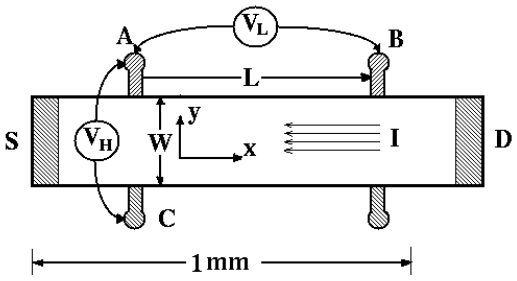
Wat is een quasideeltje? Een quasideeltje is te zien als een patroon in materie dat een lange levensduur bezit vergeleken met de processen die bestudeerd worden. Bovendien moet dit patroon zich gedragen als was het een deeltje. Hoewel veel natuurkundigen bij het woord quasideeltje niet aan een suikerkorrel denken zal ik hier suikerkorrels als voorbeeld gebruiken, omdat het aan de hand van suikerkorrels eenvoudig is om de definitie die ik net gegeven heb te illustreren. Als we willen beschrijven hoe suiker van een theelepel valt dan is het handiger om dit beschrijven met behulp van de ± 1.000 suikerkorrels dan met behulp van de $\pm 1.000.000.000.000.000.000.000$ moleculen waaruit deze suikerkorrels zijn opgebouwd. Als we echter willen beschrijven hoe suiker tijdens het roeren oplost in een kop koffie dan is het duidelijk dat de suikerkorrels niet handig zijn voor de beschrijving. De suikerkorrels verdwijnen tijdens het roeren en om te begrijpen waarom deze suikerkorrels oplossen is het beter om te gaan kijken naar de eigenschappen van de moleculen waaruit suiker is opgebouwd. Van deze analogie leren we dat als je een poging doet gedrag van materie te beschrijven het

verstandig is de juiste quasideeltjes te kiezen voor de beschrijving.
Een ander voorbeeld van quasideeltjes zijn golven op zee. Dit voorbeeld ligt al veel dichter bij het natuurkundige beeld van een quasideeltje. In een golf bewegen de watermoleculen die de echte deeltjes vormen op en neer. De golf zelf beweegt echter niet op en neer maar beweegt zich over het wateroppervlak. Dit noemt een natuurkundige collectief gedrag, want het collectief van deeltjes toont nieuw gedrag dat elk van de deeltjes afzonderlijk niet konden laten zien. Een ander voorbeeld is te zien in voetbalstadions waar de supporters de 'wave' doen. Supporters stellen nu de deeltjes voor. De golf die zij maken trekt het stadion rond maar de supporters blijven op hun plek en bewegen alleen maar op en neer.
Een andere belangrijke eigenschap van (quasi)deeljes is hun statistiek. De statistiek is een regel voor het samenvoegen van (quasi)deeltjes. Op de theelepel met suiker lagen veel suikerkorrels. Nu zouden we in principe van al deze suikerkorrels de positie en de orientatie kunnen weergeven, waarbij we voor het gemak aannemen dat alle suikerkorrels dezelfde vorm hebben. Wat we dan zouden zien is dat er nooit twee suikerkorrels op dezelfde plaats zitten. Dit gedrag heet fermionisch en correspondeert met het uitsluitingsprincipe van Pauli: elke positie kan door ten hoogste één korrel worden ingenomen. Voor golven op zee geldt een andere regel en de statistiek die zij bezitten wordt bosonisch genoemd. Verschillende golven kunnen zich te gelijkertijd op dezelfde plaats bevinden, ze kunnen door elkaar heen bewegen en ze kunnen met elkaar samengaan in een grotere golf.

N. 1.2 Fractionele Quantum Hallsystemen

Wat is een fractioneel quantum Hallsysteem? Om deze vraag te beantwoorden moeten we eerst de halfgeleider structuren (hetero-structuren) presenteren die bij zeer lage temperaturen ($20\text{mK} - 4\text{K}$) in fractionele quantum Hallsystemen te veranderen zijn door ze in een hoog magneetveld te plaatsen.
Een heterostructuur is een dubbele boterham waarbij de sneetjes brood vervangen zijn door twee verschillende halfgeleiders. Halfgeleiders vormen in de maatschappij van vandaag een belangrijke klasse van materialen, o.a. omdat ze gebruikt worden voor het bouwen van computers. De reden dat computers van halfgeleiders gemaakt worden is dat de eigenschappen van halfgeleiders goed en controleerbaar te veranderen zijn. Een zuivere halfgeleider zal bijvoorbeeld slecht stroom geleiden maar door nu de juiste onzuiverheden, electron donoren of electron acceptoren, in de halfgeleider in te brengen wordt de halfgeleider geleidend.
In een heterostructuur worden de onzuiverheden, electron donoren, dichtbij het grensvlak tussen de twee halfgeleiders geplaatst. De onzuiverheden doneren electronen aan de halfgeleider. In een normale halfgeleider structuur zouden deze electronen door de hele structuur gaan bewegen, maar in een hetero-structuur gaan deze electronen naar het grensvlak tussen de twee halfgeleiders. Door nu de heterostructuur steeds verder af te koelen zullen de electronen in hun beweging steeds meer gehinderd worden en hun beweging beperken tot een steeds

dunnere laag bij het grensvlak. Als de temperatuur voldoende laag is zitten de electronen vastgevroren aan het grensvlak. Hoewel ze nog steeds in het grensvlak kunnen bewegen zullen ze het niet meer verlaten. De wereld van deze electronen is tweedimensionaal geworden.
Wordt de heterostructuur ook nog in een sterk magneetveld gebracht dan kunnen de electronen een nieuw soort gedrag gaan vertonen. Dit gedrag is het eerst waargenomen in 1982 door Tsui, Störmer en Gossard en al snel daarna kwam Laughlin met een interpretatie van deze waarnemingen. In afbeelding N. 1.1



Figuur N. 1.1: Halfgeleider structuur met vier meetpunten , een electronen bron (S) en electronen afvoer (D). Niet getekend is het magnetische veld dat loodrecht op het grensvlak staat. De afbeelding is ontleend aan [5].

zien we de heterostructuur van boven. Aan de heterostructuur zijn vier meetpunten vastgemaakt waarvan drie een naam hebben gekregen, nl. A,B,C. Dan zijn er nog twee andere contacten: de bron (S van het engelse Source) en de afvoer (D van Drain). Vanuit de bron stromen electronen het systeem in waarna ze het via de afvoer weer verlaten. De stroom loopt in de tegengestelde richting omdat de electronen een negatieve lading hebben en de stroom richting betrekking heeft op de stroomrichting van positieve lading. De meetpunten (A,B,C) worden gebruikt om spanningsverschillen te meten. Het spanningsverschil tussen A en C heet de Hallspanning (V_H) en het spanningsverschil tussen A en B heet de longitudinale spanning (V_L).
Als er geen magneetveld aanwezig is dan is de Hallspanning nul. Is er wel een magneetveld loodrecht op het grensvlak aanwezig dan zal er een Hallspanning over de heterostructuur staan. Deze Hallspanning is het gevolg van de Lorentzkracht die op een bewegende lading in magneetveld werkt. Vanuit de klassieke

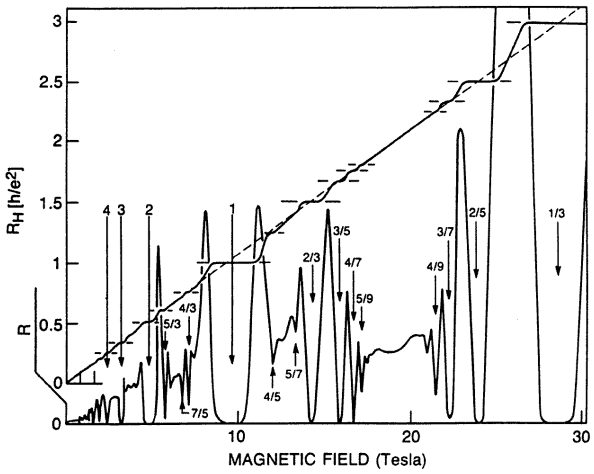
beschrijving van de Lorentzkracht verwachten we dat de Hallspanning op de volgende manier van de stroom (I) en het magneetveld (B) afhangt

$$V_H = C_{\text{Hall}} I B \tag{N. 1}$$

waarbij C_{Hall} een constante is die bekend is als de Hallcoefficient. Uit deze vergelijking kunnen we afleiden dat de Hallweerstand

$$\rho_H = \frac{V_H}{I} \tag{N. 2}$$

een lineaire afhankelijkheid van het magneetveld zal hebben. In werkelijkheid



Figuur N. 1.2: De Hallweerstand ρ_H weergegeven als functie van de magnetische veldsterkte B, afbeelding ontleend aan Willett et al.[83]

is het gedrag echter verre van lineair zoals u zelf kunt zien in afbeelding N. 1.2 waarin de resultaten van een meting van de Hallweerstand zijn getekend. In plaats van een rechte lijn zien we daar een structuur met vele plateaus. Deze plateaus bevinden zich waar de geleiding $\sigma_H = \frac{1}{\rho_H}$ een eenvoudige oneven fractie, zoals $\frac{1}{3}$ of $\frac{2}{5}$, van het fundamentele geleidings quantum $\frac{e^2}{h}$ is. Het verschijnen van deze structuur heet het fractionele quantum Halleffect waarbij de naam van het effect aangeeft dat we quantum mechanica in plaats van klassieke mechanica nodig hebben om het te begrijpen.
Nu zijn we klaar voor een definitie van een fractioneel quantum Hallsysteem: De electronen in het grensvlak tussen twee halfgeleiders in een heterostructuur vormen een fractioneel quantum Hallsysteem wanneer we ons in een meting van de Hall-weerstand op het centrum van een fractioneel quantum Halleffect plateau bevinden, waarbij de Hallgeleiding gelijk is aan een fractie van $\frac{e^2}{h}$

Het ontstaan van het fractionele quantum Halleffect, d.w.z., de formatie van plateaus en het verloop van de overgangen tussen de plateaus, wordt verklaard door elegante theorieën waarin de verschijnselen behorend bij wisselwerking op grote afstanden en die behorend bij wanorde gezamelijk beschreven worden. In dit proefschrift bespreken we deze theorieën niet, maar concentreren we ons op de materie toestanden zoals die voorkomen in fractionele quantum Hallsystemen.

N. 2 Eigenschappen van Quasideeltjes in QHS

De quasideeltjes in fractionele quantum Hall systemen bezitten twee met elkaar verweven bijzondere eigenschappen: fractionele lading en fractionele statistiek. Zoals we zullen zien is er een mooie analogie voorhanden in het bekende speelgoed Lego waarmee we deze twee begrippen kunnen illustreren.
Fractionele quantum Hall systemen bestaan uit sterk gecorreleerde electronen. Dit is onder andere zichtbaar in de manier waarop electronen uit een quantum Hall systeem naar een metalen naald in hun omgeving lekken. De Greenfunctie is de spil van de beschrijving van deze lekstroom. We zullen laten zien hoe met behulp van fractionele statistiek en nog een extra ingrediënt de Greenfunctie voor een fractioneel quantum Hallsysteem gevonden kan worden.

N. 2.1 Fractionele Lading en Statistiek

Hoewel we denken dat de kleinste vrije ladingen in het heelal de negatieve electronlading $-e$ en de positieve positronlading $+e$ zijn is het toch zo dat de quasideeltjes in fractionele quantum Hallsystemen een fractie van deze ladingen dragen. Een eenvoudig voorbeeld laat zien dat het mogelijk is om in een systeem, waarvan de bouwstenen een vaste grootte hebben, een patroon te maken waarvan de grootte een fractie is van die van de bouwstenen.
Als voorbeeld nemen we Lego-stenen met een lengte van drie noppen een breedte van één nop en een vaste hoogte. Deze stenen zetten we één laag dik op een basisplaat van één nop breed en $N \gg 1$ noppen lang. In deze rij kunnen gaten gemaakt worden die een omvang hebben van één derde van die van de Lego-stenen die we gebruiken als bouwstenen, namelijk gaten van één nop groot. Als er nu weinig gaten zijn en veel Lego-stenen dan is het handiger om te vertellen waar de gaten zitten dan te vertellen waar alle stenen staan. In een dergelijke situatie is een beschrijving in termen van quasideeltjes (de gaten) korter en efficienter. De gaten in deze nieuwe beschrijving zijn nu 'fractionele anti-stenen' en alleen als er op drie naburige plaatsen een anti-steen aanwezig is, is er voldoende ruimte om ze door een steen te vervangen. Als onze Lego-stenen een lading van $-e$ zouden bezitten dan zou het natuurlijk zijn om met elke positie op de basisplaat een lading $+e/3$ te associëren waardoor een basisplaat die volledig met stenen bedekt is geen lading zou bezitten. Met deze keuze zou een anti-steen een lading van $e/3$ bezitten. Dit is de niet gecompenseerde achtergrong lading op een positie waar geen steen aanwezig is.

De rij Lego-stenen geeft ons ook een eerste indruk van fractionele statistiek of, preciezer van *spectrum verschuivings statistiek*. Deze statistiek is het hoofdonderwerp van de hoofdstukken 'Spectral Shift Statistics' en 'Quasi-Particle Thermodynamics'. De statistiek van de stenen is direct gerelateerd aan het aantal manieren waarop je M Lego-stenen in een rij op een basisplaat van N noppen lang kan plaatsen. De Fermionische statistiek die we eerder besproken hebben zou in dit beeld overéén komen met Lego-stenen van één nop lang en de gaten en de blokken zouden dan dezelfde omvang hebben. Bosonische statistiek zou niet in dit beeld passen omdat we dan het opstapelen van meerdere Lego-stenen van één nop lang zouden moeten toestaan. Heeltallige spectrum verschuivings statistiek zou samenvallen met Fermionische statistiek behalve dat de stenen meerdere noppen g lang zouden zijn. Fractionele spectrum verschuivings statistiek wordt verkregen door van een beschrijving in termen van Lego-stenen naar een beschrijving in termen van gaten over te stappen en dan het tellen van het aantal toestanden te beschrijven uitgaande van het aantal gaten.
In de hoofdstukken waarnaar we eerder verwezen wordt grote aandacht besteed aan dualiteit. Deze dualiteit is eigenlijk niet veel meer dan zeggen dat het systeem van Lego-stenen beschreven kan worden door het geven van alle posities van de Lego-stenen of door het geven van de posities van alle gaten. Met weglating van enige belangrijke subtiliteiten kunnnen we de dualiteitsrelatie die uit deze dualiteit volgt uit onze analogie halen. Het is duidelijk dat er een relatie bestaat tussen de gatendichtheid en de Lego-stenendichtheid. Nemen we een basisplaat van N noppen lang dan zal deze lengte gelijk zijn aan het aantal gaten #(holes) plus g maal het aantal Lego-stenen #(blocks),

$$N = \#(\text{holes}) + g\ \#(\text{blocks}). \tag{N. 3}$$

Als we nu alle termen door N delen en we noemen $g\ \#(\text{holes})/N$ de gaten dichtheid $\bar{n}(\text{holes})$ en $\#(\text{blocks})/N$ de Lego-stenen dichtheid $\bar{n}(\text{blocks})$, dan krijgen we de volgende relatie,

$$g^{-1}\bar{n}(\text{holes}) = 1 - g\bar{n}(\text{blocks}). \tag{N. 4}$$

Deze relatie vertoont sterke gelijkenis met relatie 3.26 in dit proefschrift. De (normalisatie) constante die in de definitie van $\bar{n}(\text{holes})$ opduikt kan niet worden verklaard met deze analogie. In een fractioneel quantum Hall systeem kan deze constante gerelateerd worden aan het minimale energieverschil tussen twee naburige gaten. Dit energie verschil is niet $1/g$ maar $1/g^2$ maal het minimale energieverschil tussen twee naburige electronen.
In de bulk van het quantum Hall systeem wordt de rol van lengte van de basisplaat gespeeld door het aantal magnetische fluxquanta. Een electron bezet bijvoorbeeld een oppervlakte corresponderend met drie fluxquanta. Wordt het hele tweedimensionale grensvlak van de heterostructuur op deze wijze gevuld met electronen dan is het mogelijk om drie gaten te maken door een electron te verwijderen. Het is ook mogelijk om een gat te maken door een extra fluxquantum aan het systeem toe te voegen. In onze analogie zou dit overeenkomen met het verlengen van de basisplaat met één nop zonder het aantal Lego-stenen te

vergroten. Voor de rand van het quantum Hallsysteem is er geen natuurlijke N maar het blijkt dat de fractionele statistiek kan worden gedefinieerd in kleine energieintervallen die op natuurlijke wijze tot een definitie van N leiden.

N. 2.2 Verwijderen van Quasideeltjes

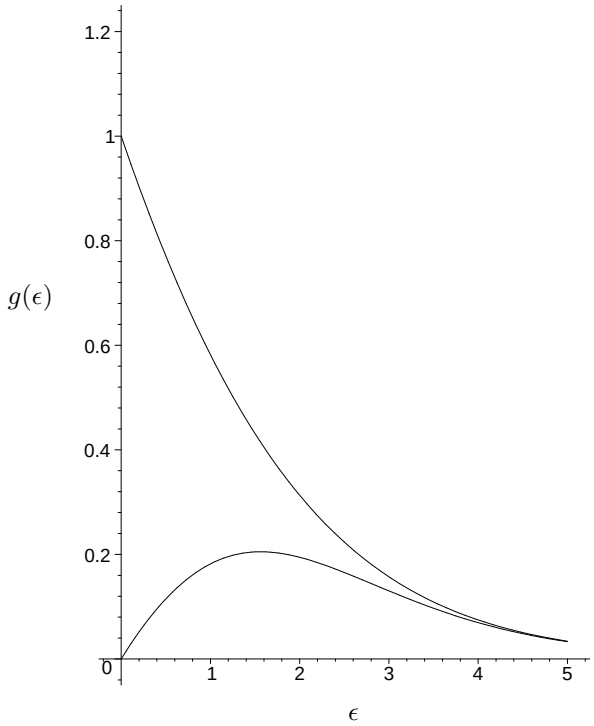
De wisselwerking tussen een fractioneel quantum Hallsysteem en de buitenwereld kan vaak worden gezien als een combinatie van processen waarbij electronen het systeem binnenkomen en/of verlaten. Een voorbeeld van een dergelijke wisselwerking is het lekken van electronen vanuit het systeem naar een metalen naald in de omgeving. Voor het berekenen van deze lekstroom gebruiken we Greenfuncties. Deze functies geven tot op een constante na het aantal electronen $g(\epsilon)\Delta\epsilon$ die in het energieinterval $[\epsilon, \epsilon + \Delta\epsilon)$ beschikbaar zijn voor deze lekprocessen. Verder hebben we voor deze berekening de waarschijnlijkheid $W(\epsilon)$ nodig waarmee een beschikbaar electron weglekt.
Wanneer het systeem waaruit de electronen weg zouden lekken zelf een metaal zou zijn, dan zou de Greensfunctie $g(\epsilon)$ eenvoudig gegeven worden door het produkt van de bezettingsgraad $\bar{n}(\epsilon)$ van een toestand bij energie ϵ maal de thermodynamische toestandsdichtheid $D(\epsilon)$,

$$g(\epsilon) = D(\epsilon)\bar{n}(\epsilon). \tag{N. 5}$$

De bijdrage van de electronen in dit interval aan de lekstroom wordt gegeven door

$$I_{\text{leakage}}([\epsilon, \epsilon + \Delta\epsilon)) \propto W(\epsilon)D(\epsilon)\bar{n}(\epsilon)\Delta\epsilon. \tag{N. 6}$$

De lekstroom hangt ook af van de Greenfunctie van de naald, maar deze afhankelijkheid hebben we in onze notatie onderdrukt om onze argumenten eenvoudig te houden.
In afbeelding N. 2.3 zien we de echte Greenfuncties en de naïeve benadering N. 5 ervan voor een specifiek fractioneel quantum Hallsysteem. Deze afbeelding laat duidelijk zien dat de naïeve benadering niet voldoet. De wisselwerkingen tussen de electronen zijn zo sterk dat de vrije deeltjes beschrijving die de achtergrond vormt voor de naïeve benadering volledig faalt. De electronen gedragen zich alsof ze aan elkaar geregen zijn. Trek je aan een electron dan komt de hele ketting van electronen mee. In werkelijkheid trekken de electronen elkaar niet aan maar stoten ze elkaar af maar het effect is hetzelfde.
Het hoofddoel van de laatste hoofdstukken van dit proefschrift is om de Greenfuncties $g(\epsilon)$ uit te drukken in grootheden zoals $D(\epsilon)$ en $\bar{n}(\epsilon)$. In plaats van $D(\epsilon)$ gebruiken wij een verzameling matrixelementen $D(\epsilon, \{\epsilon_i\}, \{\tilde{\epsilon}_j\})$ die processen beschrijven waarin een electron beschikbaar komt voor weglekken na herschikking van alle quasideeltjes die zich bij energien $\{\epsilon_i\}, \{\tilde{\epsilon}_j\}$ in het fractionele quantum Hall systeem bevinden. In het eenvoudigste (fermionische) geval is alleen het quasideeltje dat het systeem binnenkomt of verlaat van belang en is het enige matrix element betrokken bij deze herschikking $D(\epsilon)$. In dit geval kunnen we $g(\epsilon)$ ontbinden in $D(\epsilon)\bar{n}(\epsilon)$. In het algemeen is dit niet mogelijk en

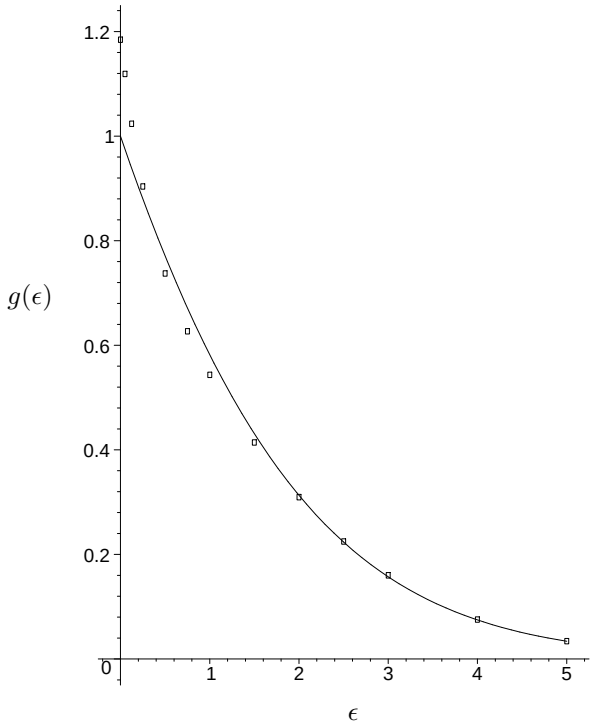


Figuur N. 2.3: De naïeve benadering $D(\epsilon)\bar{n}(\epsilon)$ (onderste kromme) van de Greenfunctie $g(\epsilon)$ (bovenste kromme)als een functie van de energie in eenheden $[k_BT]$ voor een specifiek fractioneel quantum Hall systeem.

wordt $g(\epsilon)$ verkregen uit de matrix elementen door ze te vermenigvuldigen met de distributie functies $\bar{n}_{e^-}(\epsilon_i)$ en $\bar{n}_h(\tilde{\epsilon}_j)$ en te sommeren over ϵ_i en $\tilde{\epsilon}_j$,

$$g(\epsilon) = \sum_{\{\epsilon_j\},\{\tilde{\epsilon}_i\}} D(\epsilon, \{\epsilon_i\}, \{\tilde{\epsilon}_j\})\bar{n}_{e^-}(\epsilon_1)\ldots\bar{n}_{e^-}(\epsilon_M)\bar{n}_h(\tilde{\epsilon}_1)\ldots\bar{n}_h(\tilde{\epsilon}_N). \quad \text{(N. 7)}$$

In principe zijn er oneindig veel matrixelementen, die bijdragen aan deze sommatie. Gelukkig is het zo dat we verwachten dat met een klein aantal matrixelementen, die herschikkingsprocessen van kleine aantallen quasideeltjes beschrijven, al een goede benadering van $g(\epsilon)$ kan worden verkregen.
Voor de eenvoudigste fractionele quantum Hall systemen hebben we sterke aanwijzingen gevonden dat deze verwachting klopt. In afbeelding N. 2.4 hebben we onze resultaten weergegeven. We zien dat met de naïeve benadering in afbeelding N. 2.3 de exacte functie $g(\epsilon)$ nu veel beter benaderd wordt.
Het formalisme dat is ontwikkeld in de laatste twee hoofdstukken kan worden gebruikt voor een uitbreiding van de quasi-deeltjes aanpak naar complexere fractionele quantum Hallsystemen en andere effectief ééndimensionale modellen.



Figuur N. 2.4: Greenfunctie $g(\epsilon)$: de gesloten kromme is het exacte antwoord de losse punten zijn het resultaat van onze benadering, waarin matrixelementen behorend bij maximaal twee deeltjes herschikkingsprocessen in rekening zijn gebracht.

N. 3 Nieuwe Wijsheden

In dit proefschrift hebben we aangetoond dat de fractioneel geladen quasideeltjes die op de rand van een fractioneel quantum Hallsysteem leven spectrum verschuivings statistiek bezitten. Bovendien hebben we, uitgaande van verdelingsfuncties voor quasideeltjes en de matrixelementen van de juiste fysische operatoren, laten zien dat het mogelijk is om goede benaderingen te vinden voor experimenteel relevante grootheden in eenvoudige fractionele quantum Hall systemen. Hoewel een volledig begrip van deze benadering nog ontbreekt, denken wij dat we haar validiteit hebben aangetoond. Voor uitbreiding van de in dit proefschrift gepresenteerde aanpak naar complexere fractionele quantum Hallsystemen of andere laagdimensionale sterk wisselwerkende systemen zijn verdere ontwikkelingen op het gebied van Jack-polynomen en meer algemeen symmetrische polynomen nodig. Als vooruitgang wordt geboekt op deze gebieden dan kan de quasideeltjes beschrijving in dit proefschrift verder worden gestroomlijnd

en daarmee een belangrijk stuk gereedschap worden voor de analyse van sterk gecorreleerde electron systemen.

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